Introduction to Vectors

Scalar and Vector Quantities:

Scalar

- a quantity with magnitude only.

Examples:

- Distance
- Speed
- Mass
- Temperature
- Volume
- Energy

Vector

- a mathematical quantity that is expressed by a magnitude AND direction

Examples:

- Displacement
- Weight
- Velocity
- Force
- Acceleration
- Gravity

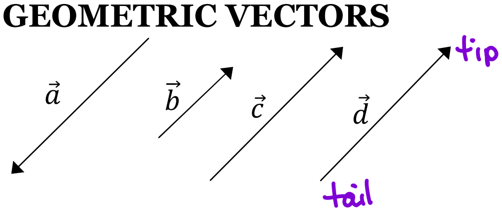
Example 1: the following situations need to be described using an appropriate measure. Classify the measure as a scalar or a vector.

(a) the cost of a dance ticket Scalar
(b) the path from your desk to the classroom door Vector
(c) the air speed of a jet as it heads north Vector

REPRESENTATION OF VECTORS:

1. GEOMETRIC VECTORS

A vector represented by a directed line segment drawn so that its length represents its magnitude.

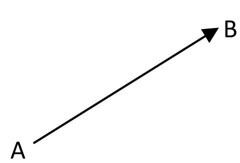


2. ALGEBRAIC VECTORS

A vector that is written in rectangular form.

- **RECTANGULAR FORM:** is of the form (a, b) or $\begin{pmatrix} a \\ b \end{pmatrix}$ or [a,b] where a represents the x-value and b represents the y-value of the terminal point of the vector. Its initial point is the origin (0, 0). They are also called **algebraic vectors** and IB uses them in the column vector form.ie. (3, 4) becomes $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Notation:



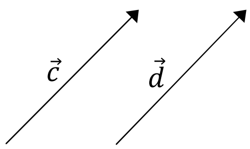
$\overrightarrow{AB}, \vec{v}$, **v (boldface)**

Vector $\overrightarrow{AB}$ has an initial point (tip) at A and a terminal point (tail) at B. (Point-to-point vector)

Magnitude of a Vector $|\overrightarrow{\boldsymbol{AB}}|$ represents the MAGNITUDE of $\overrightarrow{\boldsymbol{AB}}$.
$|\ \mathbf{v}\ |$ represents the MAGNITUDE of $\mathbf{v}$.

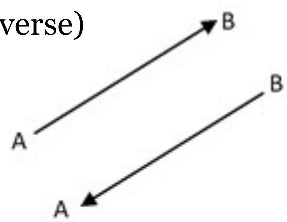
- If the magnitude of a vector is zero, we call it the zero vector and denote it $\vec{0}$. This is a useful vector despite that its direction is undefined.

Equality of Vectors



Two vectors are equal if they have the same magnitude and the same direction. Ex. $\vec{c} = \vec{d}$, $|\vec{c}| = |\vec{d}|$

Opposite Vectors
(additive inverse)

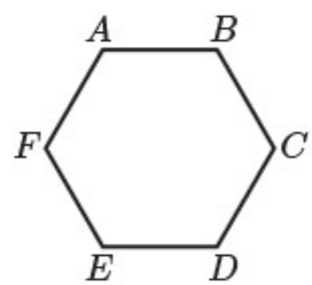


The opposite of a vector would have the same magnitude but opposite direction. Ex. $\vec{a} = -\vec{d}$, $|\vec{a}| = |\vec{d}|$

$\overrightarrow{\boldsymbol{AB}} = -\overrightarrow{\boldsymbol{BA}}$ $\therefore$ **directions are opposite**

$|\overrightarrow{\boldsymbol{AB}}| = |\overrightarrow{\boldsymbol{BA}}|$ $\therefore$ **magnitudes are still equal**

Example 2: *ABCDEF* is a regular hexagon. Give examples of vectors formed between pairs of vertices of hexagon *ABCDEF*:



a. equal

$\overrightarrow{AB} = \overrightarrow{ED}$ (or $\overrightarrow{AF} = \overrightarrow{CD}$; $\overrightarrow{FE} = \overrightarrow{BC}$; etc.)

b. parallel but with different magnitudes

$\overrightarrow{AB}$ and $\overrightarrow{FC}$ (or $\overrightarrow{AF}$ and $\overrightarrow{BE}$, etc.)

c. equal in magnitude but opposite in direction

$\overrightarrow{AB}$ and $\overrightarrow{DE}$ (or $\overrightarrow{FE}$ and $\overrightarrow{CB}$, etc.)

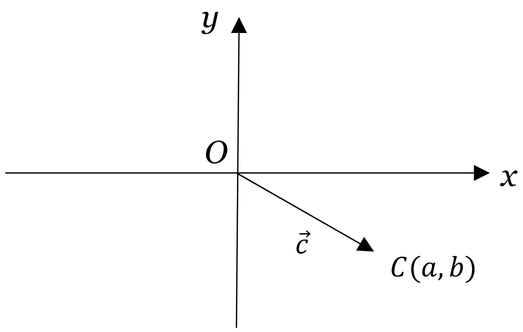
d. equal in magnitude but not parallel

$\overrightarrow{AB}$ and $\overrightarrow{BC}$ (or $\overrightarrow{FE}$ and $\overrightarrow{CD}$, etc)

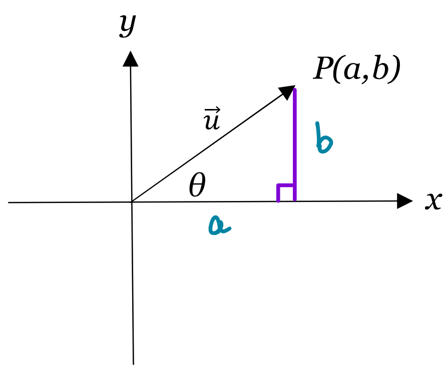
e. different in both magnitude and direction

$\overrightarrow{AB}$ and $\overrightarrow{CE}$

Note: vector starting from the origin O to another point C is called position vector of C (i.e., fixed with respect to the origin). $\overrightarrow{OC} = \vec{c}$ or $\overrightarrow{OC} = \binom{a}{b}$ ‘column vector’

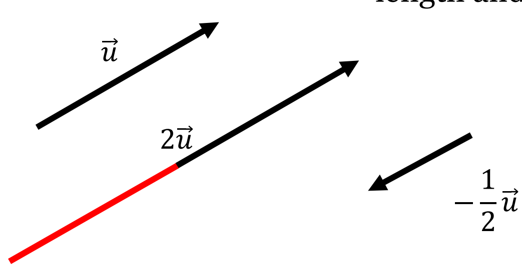


Cartesian Co-ordinate System: $\vec{u}$ can be represented as an ordered pair (a, b) where its magnitude (modulus) is $|\vec{u}| = \sqrt{a^2 + b^2}$ and direction $\theta = tan^{-1}\left(\frac{b}{a}\right)$ with θ measured counter-clockwise from the positive x-axis to the line of the vector. The ordered pair (a, b) is referred to as an **algebraic vector.** The values of a and b are the x- and y-components of the vector.

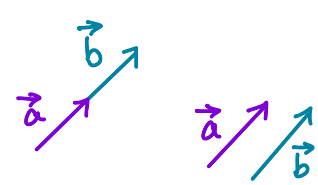


Scalar Multiplication:

Multiplying a vector by a scalar will change its length and/or its direction



Collinear Vectors



- vectors that lie on the same line when they are in standard position.
- they will be parallel to each other.
- they either have the same direction or opposite direction.
- one will always be a scalar multiple of the other. i.e. $\boldsymbol{\vec{u} = k\vec{v}, k \in R, k \neq 0}$

Example 3. Given that $|\vec{u}| = 5$ find the magnitude of each of the following vectors:

a. $2\vec{u}$

$|2\vec{u}| = 2|\vec{u}|$
$= 2(5)$
$= 10$

b. $-4\vec{u}$

$|-4\vec{u}| = 4|\vec{u}|$
$= 4(5)$
$= 20$

c. $\frac{1}{5}\vec{u}$

$|\frac{1}{5}\vec{u}| = \frac{1}{5}|\vec{u}|$
$= \frac{1}{5}(5)$ $= 1$

Example 4. Determine the value of k so that the following pairs of vectors are collinear.

a. $\begin{pmatrix} 5 \\ -7 \end{pmatrix} = k\begin{pmatrix} -10 \\ 14 \end{pmatrix}$

$5 = -10k$ or $-7 = 14k$
$-\frac{1}{2} = k$ $\quad -\frac{1}{2} = k$

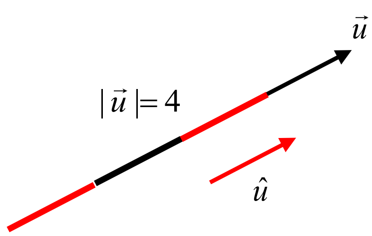
b. $\vec{v} = \left(1, \frac{k}{2} - 6\right)$ and $\vec{u} = \left(-4, 1 + \frac{k}{6}\right)$

$\vec{v} = n\vec{u}$
$\therefore 1 = -4n$
$-\frac{1}{4} = n$

$\therefore \frac{k}{2} - 6 = \frac{-1}{4}\left(1 + \frac{k}{6}\right)$
$\frac{k}{2} - 6 = -\frac{1}{4} - \frac{k}{24}$
$\frac{13k}{24} = \frac{23}{4}$
$k = \frac{552}{52}$
$= \frac{138}{13}$

The Unit Vector:

A unit vector is a vector 1 unit long.



$\hat{u}$ is a unit vector parallel to $\vec{u}$

$\hat{u} = \frac{1}{|\vec{u}|}\vec{u}$ or $\hat{u} = \frac{\vec{u}}{|\vec{u}|}$

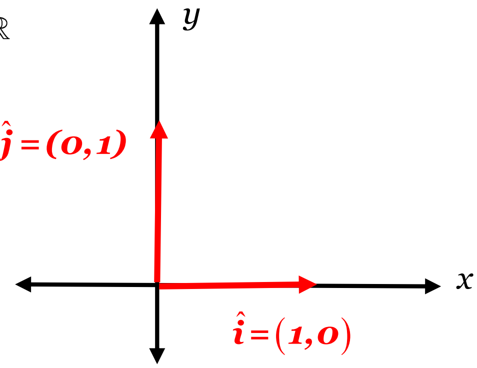
To create a unit vector in the direction of a non-zero $\vec{u}$, multiply $\vec{u}$ by the scalar equal to the reciprocal of the magnitude of $\vec{u}$. or divide $\vec{u}$ by its magnitude $|\vec{u}|$

Example 5: If $|\vec{a}| = 12$, state a unit vector in the opposite direction of $\vec{a}$.

$\hat{a} = \frac{-1}{|\vec{a}|}\vec{a}$
$= -\frac{1}{12}\vec{a}$

Standard Unit Vectors: The special unit vectors which point in the direction of the positive x-axis and positive y-axis are given the names $\hat{i}$ and $\hat{j}$ respectively, where $\hat{i} = (1,0)$ and $\hat{j} = (0,1)$.

$\hat{i}$ and $\hat{j}$ are the ***standard basis vectors*** for $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$



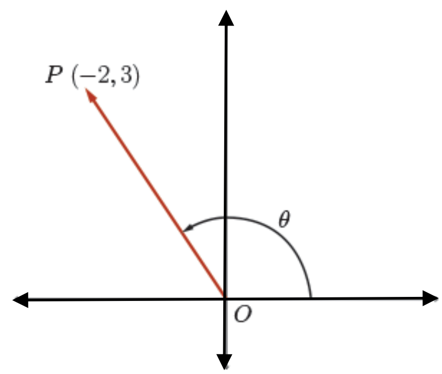
We may express any vector in the *xy*-plane as a sum of scalar multiples of the vectors and.

$\vec{u} = \overrightarrow{OP} = (a,b)$ or $\vec{u} = a\hat{i} + b\hat{j}$ or $\vec{u} = \begin{pmatrix} a \\ b \end{pmatrix}$, $|\vec{u}| = \sqrt{a^2 + b^2}$

or $[a, b]$

Example 6 : $\overrightarrow{OP} = (-2,3) = -2\hat{i} + 3\hat{j}$

$$|\overrightarrow{OP}| = \sqrt{(-2)^2 + (3)^2}$$
$$= \sqrt{13}$$
$$\tan(\theta) = \frac{3}{-2} \Rightarrow \alpha = \tan^{-1}\left(\frac{3}{2}\right) \approx 56.3^0$$
$$\theta \approx 180^0 - 56.3^0 \approx 123.7^0$$



Example 7: Find a vector of magnitude $\sqrt{6}$ in the direction of $\vec{v} = 7\hat{i} + 5\hat{j}$.

$$|\vec{v}| = \sqrt{(7)^2 + (5)^2}$$
$$= \sqrt{74}$$
$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$
$$\hat{v} = \frac{7\hat{i} + 5\hat{j}}{\sqrt{74}}$$
$$= \frac{7\hat{i}}{\sqrt{74}} + \frac{5\hat{j}}{\sqrt{74}}$$

Let the vector of magnitude $\sqrt{6}$ be m:

$$\vec{m} = \sqrt{6}\,\hat{v}$$
$$= \sqrt{6}\left(\frac{7\hat{i}}{\sqrt{74}} + \frac{5\hat{j}}{\sqrt{74}}\right)$$
$$= \frac{7\sqrt{3}\,\hat{i}}{\sqrt{37}} + \frac{5\sqrt{3}\,\hat{j}}{\sqrt{37}}$$
$$= \frac{7\sqrt{111}\,\hat{i}}{37} + \frac{5\sqrt{111}\,\hat{j}}{37}$$

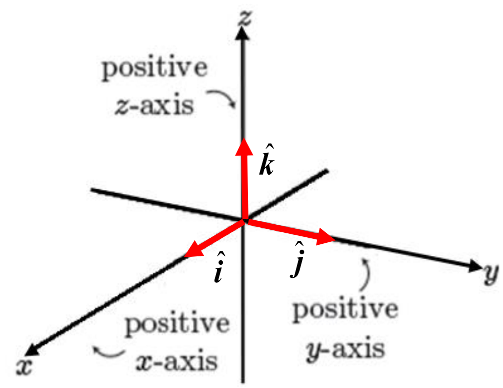
Vectors in 3 Dimensions

Previously, we had considered geometric and algebraic vectors in the 2-dimensional (Cartesian) plane. This model extends naturally to 3 dimensions.

Consider the 2-D (*xy*) Cartesian plane comprised of the *x* and *y*-axes.

If we add a third axis (*z*-axis) to our existing *xy*-plane such that all 3 axes are mutually perpendicular to one another, we create a coordinate system which models 3-dimensional space.

To plot the 3-dimensional point with coordinates (*a*,*b*,*c*), move *a* units from the origin in the *x*-direction, *b* units in the *y*-direction, and *c* units in the *z*-direction.



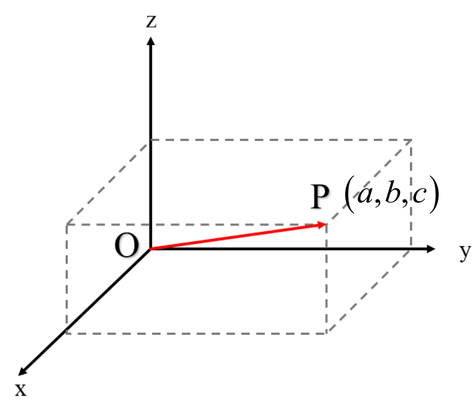
"right-handed coordinate system"

Standard Unit Vectors in R3: $\hat{\text{i}} = (1,0,0), \hat{\text{j}} = (0,1,0),$ and $\hat{\text{k}} = (0,0,1)$ are the special unit vectors pointing in the direction of the positive x-, y-, and z-axes, respectively.

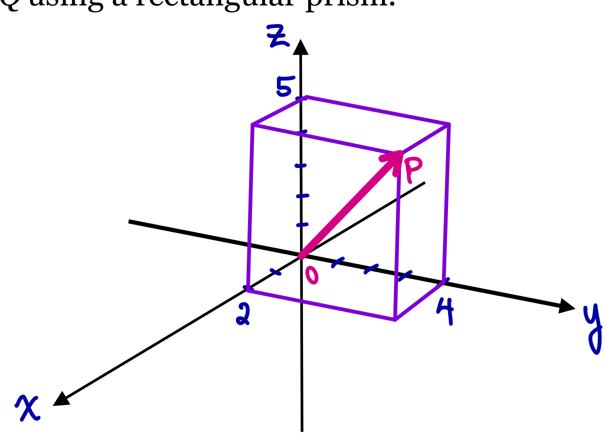
$$\overrightarrow{OP} = a\hat{i} + b\hat{j} + c\hat{k}$$

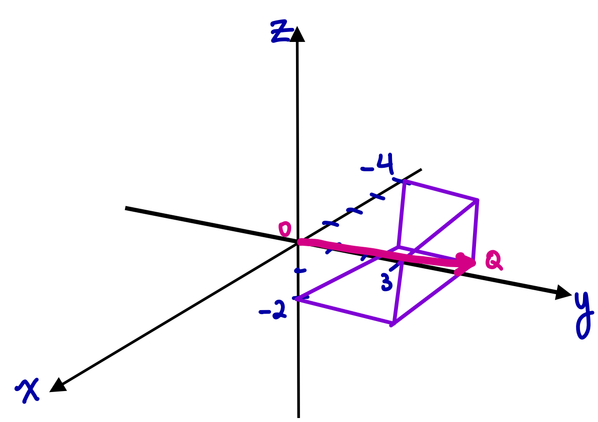
$$\overrightarrow{OP} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$|\overrightarrow{OP}| = \sqrt{a^2 + b^2 + c^2}$$



Example 8: Given the coordinates of points P(2,4,5) and Q(-4,3,-2), draw the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ using a rectangular prism.



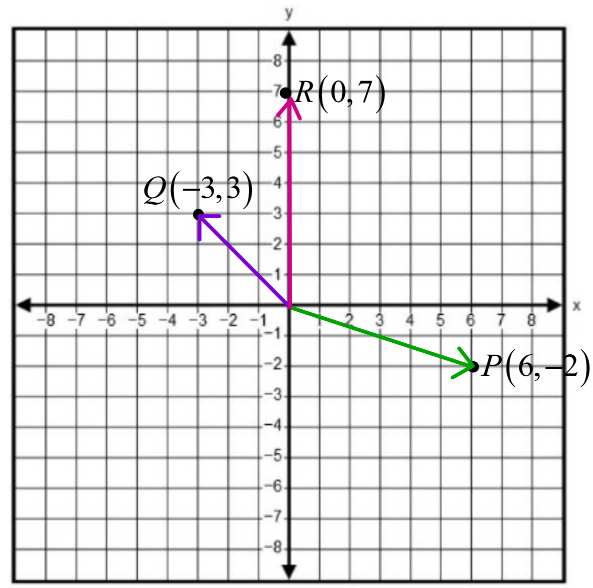


Example 9: Express the position vector of each of the points shown in the diagram as an ordered pair, column vector, and in basis vector notation.

$\vec{OQ} = (-3,3)$
$= \begin{pmatrix} -3 \\ 3 \end{pmatrix}$
$= -3\hat{i} + 3\hat{j}$

$\vec{OR} = (0,7)$
$= \begin{pmatrix} 0 \\ 7 \end{pmatrix}$
$= 7\hat{j}$

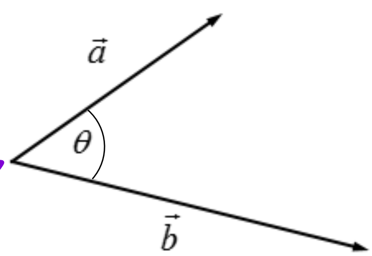
$\vec{OP} = (6,-2)$
$= \begin{pmatrix} 6 \\ -2 \end{pmatrix}$
$= 6\hat{i} - 2\hat{j}$



Angle between 2 vectors:

The angle between two vectors is the angle ≤180° formed when the vectors are placed **tail to tail**, that is, starting at the same point.

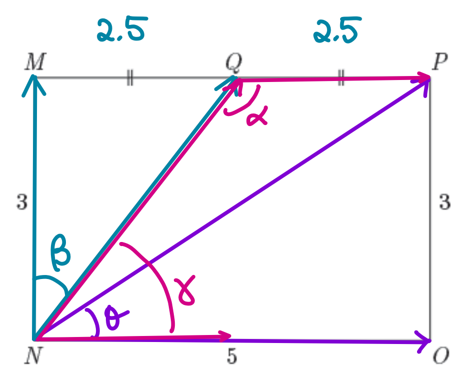
tail to tail



Example 10: *MNOP* is a rectangle with side lengths 3 and 5.

Q is the midpoint of *MP*.

Find the angle between the following vectors:



a) $\vec{NP}$ and $\vec{NO}$

$\tan(\theta) = \frac{3}{5}$
$\theta = \tan^{-1}\left(\frac{3}{5}\right)$
$\theta \doteq 31.0°$

b) $\vec{NM}$ and $\vec{NQ}$

$\tan(\beta) = \frac{2.5}{3} \rightarrow \beta = \tan^{-1}\left(\frac{2.5}{3}\right)$
$\beta \doteq 39.8°$

c) $\vec{NQ}$ and $\vec{QP}$

$|\vec{NQ}| = \sqrt{3^2 + (2.5)^2}$
$= \sqrt{15.25}$
$\doteq 3.91$

$|\vec{NP}| = \sqrt{3^2 + 5^2}$
$= \sqrt{34}$

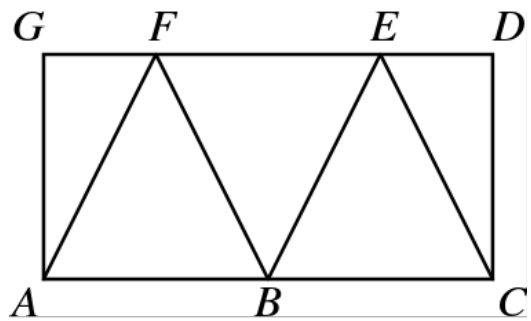
$\cos(\alpha) = \frac{(\sqrt{34})^2 - (3.91)^2 - (2.5)^2}{-2(2.5)(3.91)}$

$\alpha \doteq 129.6°$

$\gamma \doteq 180° - 129.6°$
$\doteq 50.4°$

Practice

1. In the diagram at the right, △ AF B and △ BEC are equilateral, and ACDG is a rectangle.



(a) Write down two other vectors **equal** to $\overrightarrow{AB}$.
(b) Write down three vectors which are **opposite** to $\overrightarrow{FE}$
(c) What vector is the **opposite** of $\overrightarrow{DC}$?
(d) Write down 3 vectors which have the same magnitude as $\overrightarrow{BC}$, but different direction.
(e) What vector is equal to $2\overrightarrow{FE}$?
(f) What vector is equal to $\frac{1}{2}\overrightarrow{FE}$?

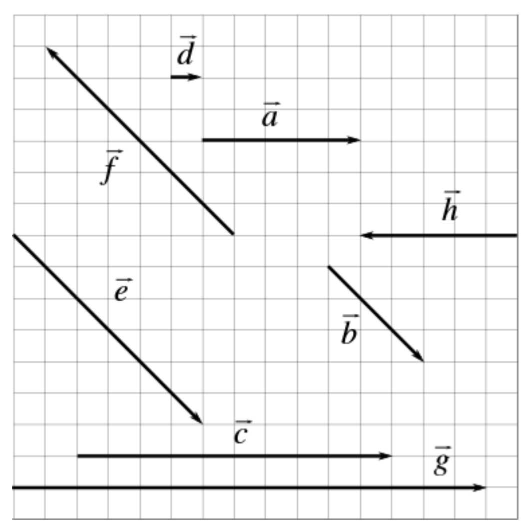
2. Using the diagram from #1, find the angles between the following vectors:

(a) $\overrightarrow{AB}$ and $\overrightarrow{AF}$
(b) $\overrightarrow{AF}$ and $\overrightarrow{AG}$
(c) $\overrightarrow{DC}$ and $\overrightarrow{AB}$
(d) $\overrightarrow{BC}$ and $\overrightarrow{CE}$
(e) $\overrightarrow{EC}$ and $\overrightarrow{AG}$
(f) $\overrightarrow{FD}$ and $\overrightarrow{BA}$

3. Sketch a vector to represent each of the following quantities, using the specified scale:

(a) a velocity of 30 m/s [south], where 1 cm = 10 m/s.
(b) a force of 20 Newtons, straight down, where 1 cm = 10 N .
(c) a displacement of 25 metres to the right, where 1 cm = 10 m.
(d) an airplane taking off a runway at an angle of 30° at a speed of 40 km/h, where 1 cm =10 km/h.

4. Using the grid at the right, choose a vector which equals:



(a) $-\vec{a}$
(b) $3\vec{a}$
(c) $-2\vec{b}$
(d) a unit vector parallel to $\vec{a}$

5. Given the vector $\vec{u}$ such that $|\vec{u}| = 8$ units, find the following:

(a) $|3\vec{u}|$
(b) $\left|-\frac{3}{4}\vec{u}\right|$
(c) $|-7\vec{u}|$

6. Determine a unit vector parallel to each of the following vectors:
   (a) $\vec{a}$, given that $|\vec{a}| = 12$ units
   (b) $\vec{w}$, given that $|\vec{w}| = 10$ units
   (c) $\vec{u}$ (non-zero)

7. A boat leaves harbour at 2:00 and travels due south at 50 km/h until 3: 30, when it turns east and travels at the same speed for another hour.

   (a) Write down the displacement vectors for each part of the journey.
   (b) What is the total distance covered?
   (c) What is the displacement vector between the starting point and ending point?

8. Two planes leave an airport at the same time. Plane A travels northwest at 120 km/h, while plane B travels due east at 150 km/h. After one hour, they both land. If plane A must then travel to plane B's landing point, in what direction should it travel, and how long will it take if it travels at 120 km/h?

9. For each point Q given, write the position vector $\overrightarrow{OQ}$ in terms of $\hat{i}$ and $\hat{j}$.

a. Q (3, −4)

b. Q (−5, −1)

10. For each point Q in question 1, find the magnitude of the position vector $\overrightarrow{OQ}$ and its direction relative to the positive x-axis.

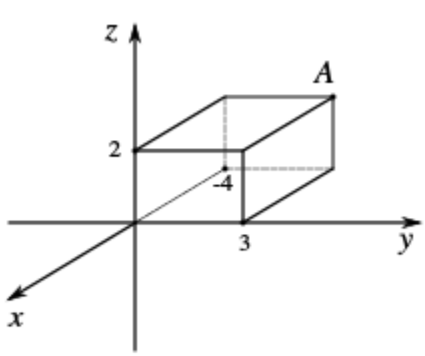
11. For each point R given, find the magnitude of the position vector $\overrightarrow{OR}$.

a. R (4, −3, 12)

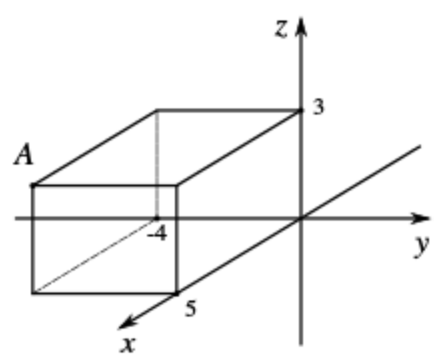
b. R (2, −1, 3)

12. Write the position vectors of the point A shown, in the form $a\hat{i} + b\hat{j} + c\hat{k}$.

a.



b.



13. Draw a sketch to show the point D (4, 2, −3) and draw the position vector $\overrightarrow{OD}$.

14. Determine the direction angles for each of the following vectors.

a. $\vec{v} = 2\hat{i} - \hat{j} + 3\hat{k}$

b. $\overrightarrow{OA} = (-1, 4, -5)$

c. $\vec{u} = 5\hat{i} - 12\hat{k}$

d. $\overrightarrow{OB} = (0, 3, -4)$

15. Find a unit vector parallel to each of the given vectors.

a. $\vec{v} = (2, -5)$

b. $\overrightarrow{OZ} = \hat{i} - 2\hat{j} + 4\hat{k}$

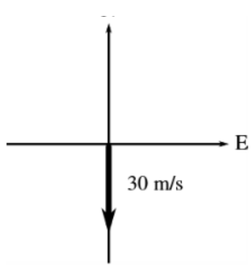
c. $\vec{w} = (-5, 12)$

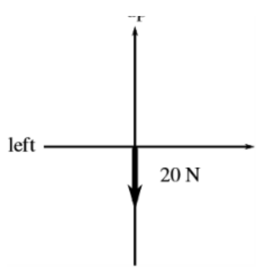
d. $\overrightarrow{OP} = 3\hat{i} + 3\hat{j} - \hat{k}$

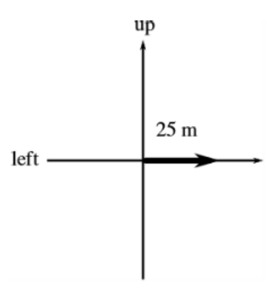
Answers

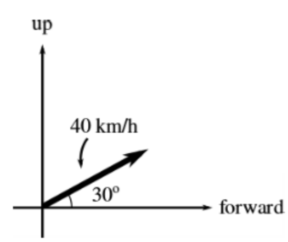
1. a. $\overrightarrow{BC}$ and $\overrightarrow{FE}$
   b. $\overrightarrow{BA}, \overrightarrow{CB}$ and $\overrightarrow{EF}$
   c. $\overrightarrow{CD}$ or $\overrightarrow{AG}$
   d. Any of $\overrightarrow{CB}, \overrightarrow{BA}, \overrightarrow{EF}, \pm\overrightarrow{AF}, \pm\overrightarrow{FB}, \pm\overrightarrow{BE}, \pm\overrightarrow{CE}$
   e. $\overrightarrow{AC}$
   f. $\overrightarrow{GF}$ or $\overrightarrow{ED}$

2. a. 60°
   b. 30°
   c. 90°
   d. 120°
   e. 150°
   f. 180°

3. a.


b.


c.


d.


4. a. $\vec{h}$
   b. $\vec{g}$
   c. $\vec{f}$
   d. $\vec{d}$

5. a. 24
   b. 6
   c. 56

6. a. $a = \frac{\vec{a}}{12}$
   b. $w = \frac{\vec{w}}{10}$
   c. $u = \frac{\vec{u}}{|\vec{u}|}$

7. a. $75\ \mathrm{km}[S], 50\ \mathrm{km}[E]$
   b. $125\ \mathrm{km}$
   c. $90.1\ \mathrm{km}[S33.7^\circ E]$

8. $[E\ 20^\circ\ S]$ for 2 hours and 5 minutes

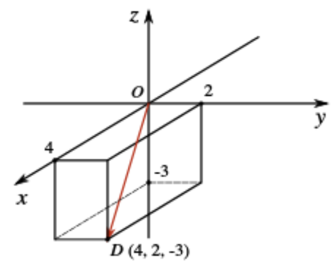
9. a. $3\hat{i} - 4\hat{j}$
   b. $-5\hat{i} - \hat{j}$

10. a. $\left|\overrightarrow{OQ}\right| = 5, 307^\circ$ or -53° from the positive x-axis.
    b. $\left|\overrightarrow{OQ}\right| = \sqrt{26}, 191^\circ$ or -169° from the positive x-axis.

11. a. $\left|\overrightarrow{OR}\right| = 13$
    b. $\left|\overrightarrow{OR}\right| = \sqrt{14}$

12. a. $\overrightarrow{OA} = -4\hat{i} + 3\hat{j} + 2\hat{k}$
    b. $\overrightarrow{OA} = 5\hat{i} - 4\hat{j} + 3\hat{k}$

13. Answers may vary.



14. $\vec{v} \approx 5\hat{i} - 2\hat{j} - 4\hat{k}$

15. a. $\left(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\right)$
    b. $\frac{1}{\sqrt{21}}\hat{i} - \frac{2}{\sqrt{21}}\hat{j} + \frac{4}{\sqrt{21}}\hat{k}$
    c. $\left(-\frac{5}{13}, \frac{12}{13}\right)$
    d. $\frac{3}{\sqrt{19}}\hat{i} + \frac{3}{\sqrt{19}}\hat{j} - \frac{1}{\sqrt{19}}\hat{k}$

Sample Questions

1. If $\vec{v}$=(-2, 7) then find the magnitude of $\vec{v}$ and the angle it makes with the x-axis.

2. If $\vec{v}$=(-3, 2, 5) then what is the vector going in the opposite direction to $\vec{v}$ that has a magnitude of 12?

3. If a wind, $\vec{v}$, is moving 15 km/h in the direction of $S60°E$ then determine $\vec{v}$ in component form. Hint: Make a diagram with a right angled triangle *w* as the hypotenuse and use trig to solve for the length on the x-and y-axis to find the x- and y- components of $\vec{v}$

Sample Questions

1. If $\vec{v}$ =(-2, 7) then find the magnitude of $\vec{v}$ and the angle it makes with the x-axis.

$\vec{V} = \sqrt{(-2)^2 + 7^2}$

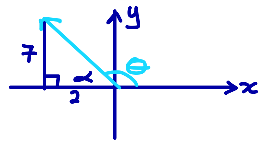
$\vec{V} = \sqrt{53}$

$\alpha = \tan^{-1}\left(\frac{7}{2}\right)$

$\alpha = 74.05°$

$\theta = 180° - 74.05°$

$\theta = 106°$



2. If $\vec{v}$ =(-3, 2, 5) then what is the vector going in the opposite direction to $\vec{v}$ that has a magnitude of 12?

$|\vec{v}| = \sqrt{(-3)^2 + 2^2 + 5^2}$

$|\vec{v}| = \sqrt{38}$

$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$

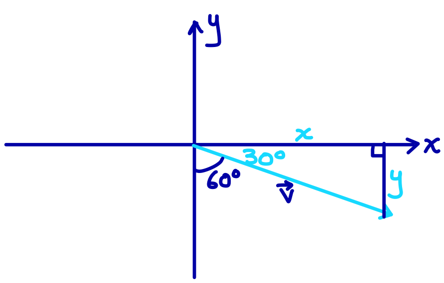
$= \frac{[-3, 2, 5]}{\sqrt{38}}$

Let $\vec{m}$ has a magnitude of 12, going in the opposite direction to $\vec{v}$

$\vec{m} = 12\frac{[3, -2, -5]}{\sqrt{38}}$

$\vec{m} = \left[\frac{36}{\sqrt{38}}, \frac{-24}{\sqrt{38}}, \frac{-60}{\sqrt{38}}\right]$

3. If a wind, $\vec{v}$, is moving 15 km/h in the direction of $S60°E$ then determine $\vec{v}$ in component form. Hint: Make a diagram with a right angled triangle *w* as the hypotenuse and use trig to solve for the length on the x-and y-axis to find the x- and y- components of $\vec{v}$



$|\vec{v}| = 15$

$\cos 30° = \frac{x}{|\vec{v}|}$

$x = \left(\frac{\sqrt{3}}{2}\right)(15)$

$x = \frac{15\sqrt{3}}{2}$

$\sin 30° = \frac{y}{|\vec{v}|}$

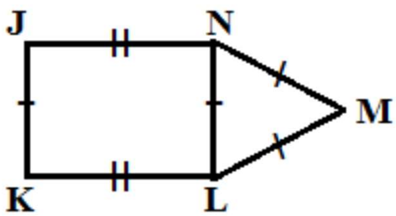
$y = \frac{1}{2}(15)$

$y = \frac{15}{2}$

$\vec{v} = \left[\frac{15\sqrt{3}}{2}, \frac{15}{2}\right]$

Warm-up: Introduction to Vectors

1. Name all equal vectors in the diagram.

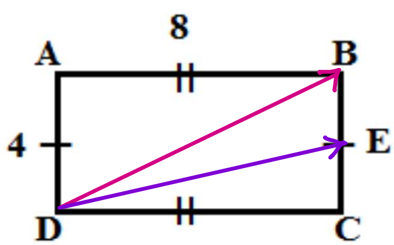


$\vec{JN} = \vec{KL}$ or $\vec{NJ} = \vec{LK}$

$\vec{JK} = \vec{NL}$

2. ABCD is a rectangle with sides measuring 8 units and 4 units. E is the midpoint of BC. Find the angle between the following vectors:

a) $\overrightarrow{DB}$ and $\overrightarrow{DC}$ b) $\overrightarrow{DE}$ and $\overrightarrow{DC}$ c) $\overrightarrow{DB}$ and $\overrightarrow{DE}$



$\tan(\angle BDC) = \frac{4}{8}$

$\angle BDC = \tan^{-1}\left(\frac{1}{2}\right)$

$\angle BDC \doteq 26.6°$

$\tan(\angle EDC) = \frac{2}{8}$

$\angle EDC = \tan^{-1}\left(\frac{1}{4}\right)$

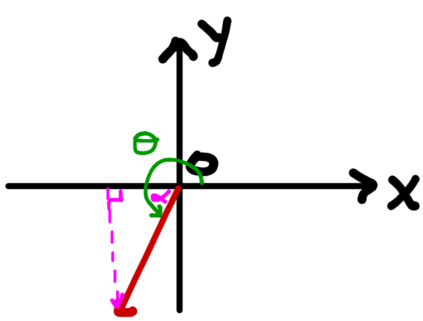
$\angle EDC \doteq 14.0°$

$\angle BDE = \angle BDC - \angle EDC$

$= 26.6° - 14.0°$

$= 12.6°$

3. Draw the position vector of the point P (-2,-5). Express it in ordered pair notation, basis vector notation and column notation. Determine its magnitude and direction.



P(-2,-5)

$\vec{OP} = [-2,-5]$

$= -2\hat{i} - 5\hat{j}$

$= \begin{pmatrix} -2 \\ -5 \end{pmatrix}$

$|\vec{OP}| = \sqrt{(-2)^2 + (-5)^2}$

$= \sqrt{29}$

direction : $\tan\alpha = \frac{5}{2}$

$\alpha = \tan^{-1}\left(\frac{5}{2}\right)$

$\alpha \approx 68.2°$ "related acute ∠"

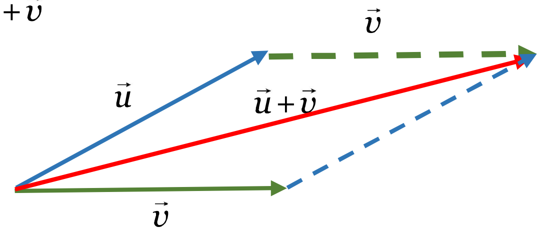
$\therefore \theta = 180° + 68.2°$

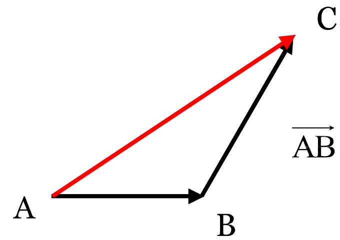
$\doteq 248°$

$\therefore$ The magnitude of $\vec{OP}$ is $\sqrt{29}$ units and its direction is 248° with the positive x-axis.

Vector Laws:

Triangular Law of Vector Addition: For vectors $\vec{u}$ and $\vec{v}$, the sum (or resultant) of $\vec{u}$ and $\vec{v}$ is a vector from the tail of $\vec{u}$ to the tip of $\vec{v}$, when the tail of $\vec{v}$ is placed at the tip of $\vec{u}$.
Notation: $\vec{u}+\vec{v}$





$\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

Commutative Law of Addition: $\boxed{\vec{u}+\vec{v}=\vec{v}+\vec{u}}$

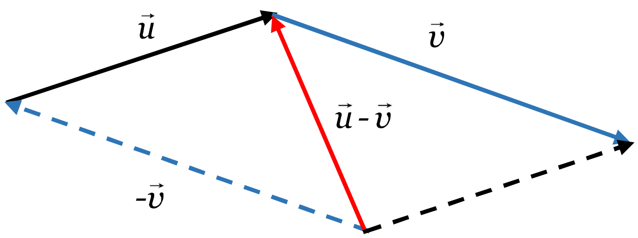
Note: sum of the vectors is the **diagonal** of the parallelogram (**Parallelogram Law of Vector Addition**)

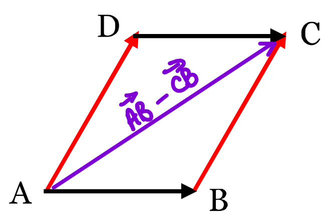
Recap

The **triangle law** was useful when arranging vectors tip to tail.

The **parallelogram law** was useful when arranging vectors tail to tail.

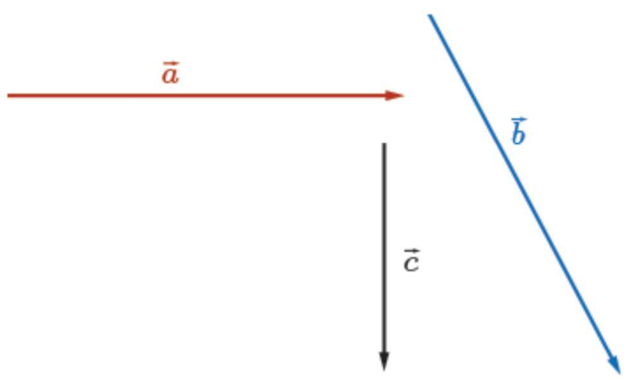
Vector Subtraction (adding the opposite): For vectors $\vec{u}$ and $\vec{v}$, $\boxed{\vec{u}-\vec{v}=\vec{u}+(-\vec{v})}$

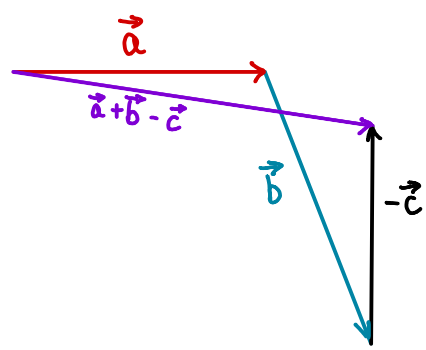




Or $\overrightarrow{AB}-\overrightarrow{CB}=\overrightarrow{AB}+\overrightarrow{BC}$

Example 1: Determine $\vec{a}+\vec{b}-\vec{c}$.





Vector Operations	2-dimensions	3-dimensions
• **Vector addition**	$\vec{u}+\vec{v}=(u_1+v_1, u_2+v_2)$	$\vec{u}+\vec{v}=(u_1+v_1, u_2+v_2, u_3+v_3)$
• **Vector subtraction**	$\vec{u}-\vec{v}=(u_1-v_1, u_2-v_2)$	$\vec{u}-\vec{v}=(u_1-v_1, u_2-v_2, u_3-v_3)$
• **Scalar multiplication:**	$k\vec{u}=(ku_1, ku_2)$	$k\vec{u}=(ku_1, ku_2, ku_3)$

Components of a vector between two points:

The points $A(x_1, y_1)$ and $B(x_2, y_2)$ form the vector $\overrightarrow{AB}$. Using "position vectors", determine $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$. [Hint: To do this, use the triangle law of addition.]

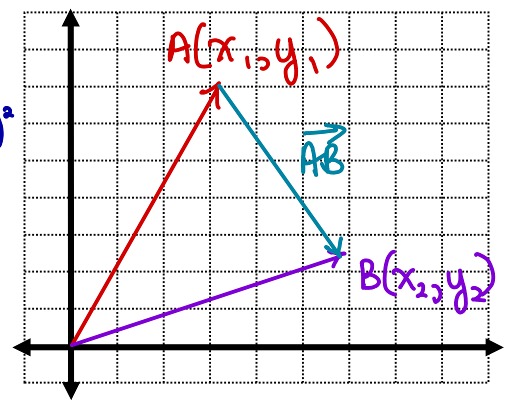
$\vec{OA}+\vec{AB}=\vec{OB}$

$\therefore \vec{AB}=\vec{OB}-\vec{OA}$

$=[x_2, y_2]-[x_1, y_1]$

$=[x_2-x_1, y_2-y_1]$

$|\vec{AB}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$



$\boxed{\therefore \vec{AB}=[x_2-x_1, y_2-y_1]}$ ← Point-to-point vector

Example 2 :P(4, 5), Q(-7, 10) and R(8, -3) are three points in R².

a) Determine $\overrightarrow{QP}$ and $|\overrightarrow{QP}|$.

$\vec{QP}=\vec{QO}+\vec{OP}$

$=\vec{OP}-\vec{OQ}$

$=[4,5]-[-7,10]$

$=[11,-5]$

$|\vec{QP}|=\sqrt{11^2+(-5)^2}$

$=\sqrt{146}$

b) Determine $|\overrightarrow{PQ}+\overrightarrow{QR}|$.

$\vec{PQ}+\vec{QR}=[-7-4, 10-5]+[8-(-7), -3-10]$

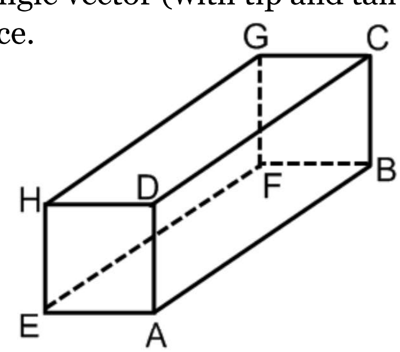
$=[-11,5]+[15,-13]$

$=[4,-8]$

$|\vec{PQ}+\vec{QR}|=\sqrt{(4)^2+(-8)^2}$

$=\sqrt{80} \Rightarrow 4\sqrt{5}$

Example 3: The diagram shows a rectangular prism. Determine a single vector (with tip and tail on the rectangular prism) that is equivalent to each sum or difference.



a) $-\overrightarrow{AE}+\overrightarrow{DA}=$ $\vec{EA}+\vec{DA}=\vec{HD}+\vec{DA}=\vec{HA}$

b) $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CG}=$ $\vec{AB}+\vec{BC}+\vec{CG}=\vec{AG}$

c) $\overrightarrow{FG}-\overrightarrow{DC}-\overrightarrow{AE}+\overrightarrow{AF}=$ $\vec{FG}+\vec{CD}+\vec{EA}+\vec{AF}$

$=\vec{FG}+\vec{GH}+\vec{HD}+\vec{DG}$

$=\vec{FG}$

or $\vec{FG}+\vec{GH}+\vec{EF}$

$=\vec{FH}+\vec{EF}$

$=\vec{EH}$

Example 4: Given the vectors $\vec{u} = (2,4,-1)$, $\vec{v} = 5\hat{\imath} + 4\hat{k}$ and $\vec{w} = (-1,3,5)$, determine the following:

a) $2\vec{u} - \vec{w}$

$= 2[2,4,-1] - [-1,3,5]$

$= [5,5,-7]$

b) $\vec{u} - \frac{\vec{v}}{2}$

$= [2,4,-1] - \frac{1}{2}[5,0,4]$

$= [-\frac{1}{2}, 4, -3]$

c) $|\vec{u} + \vec{v} - \vec{w}|$

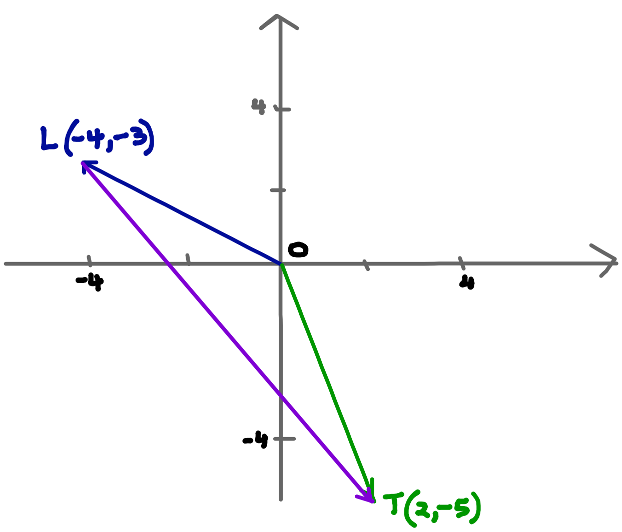
$\vec{u} + \vec{v} - \vec{w} = [2,4,-1] + [5,0,4] - [-1,3,5]$

$= [8,1,-2]$

$|\vec{u} + \vec{v} - \vec{w}| = \sqrt{8^2 + 1^2 + (-2)^2}$

$= \sqrt{69}$

Example 5: A surveyor is standing at the top of a hill. Call this point the origin O. A lighthouse, L, is visible 4 km to the west and 3 km to the north. A town, T, is visible 5 km to the south and 2 km to the east. Using a vector basis in which $\hat{i}$ is a 1 km vector running east and $\hat{j}$ is 1 km vector running north, the position vectors of the lighthouse, $\overrightarrow{OL}$ and the town $\overrightarrow{OT}$. Hence, find the vector $\overrightarrow{LT}$ and the position of the town relative to the lighthouse.



$\overrightarrow{OL} = -4\hat{\imath} + 3\hat{\jmath}$

$\overrightarrow{OT} = 2\hat{\imath} - 5\hat{\jmath}$

$\overrightarrow{LT} = \overrightarrow{LO} + \overrightarrow{OT}$

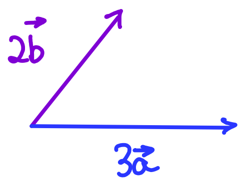
$= -\overrightarrow{OL} + \overrightarrow{OT}$

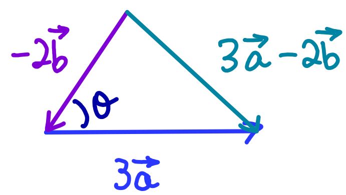
$= -(-4\hat{\imath} + 3\hat{\jmath}) + (2\hat{\imath} - 5\hat{\jmath})$

$= 6\hat{\imath} - 8\hat{\jmath}$

$\therefore$ The town is 6 km east and 8 km south of the lighthouse.

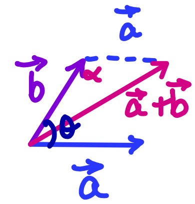
Example 6: Determine the value of $|\vec{a}+\vec{b}|$ if $|3\vec{a}| = 24\text{cm}$,$|2\vec{b}| = 10\text{cm}$ and $|3\vec{a}-2\vec{b}| = 20\text{cm}$.





$\because |3\vec{a}| = 24$ $\quad$ $\because |2\vec{b}| = 10$

$\therefore |\vec{a}| = 8$ $\quad$ $\therefore |\vec{b}| = 5$



$\alpha \doteq 180^\circ - 54.9^\circ$

$\doteq 125.1^\circ$

$$\cos(\theta) = \frac{(20)^2-(10)^2-(24)^2}{-2(10)(24)}$$

$\theta \doteq 54.9^\circ$

$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\alpha)$

$|\vec{a}+\vec{b}|^2 = 8^2 + 5^2 - 2(8)(5)\cos(125.1^\circ)$

$|\vec{a}+\vec{b}| \doteq 11.6 \text{ cm}$

Example 7: Given that

$\vec{u} = x\vec{a} + 2y\vec{b}$

$\vec{v} = -2y\vec{a} + 3y\vec{b}$

$\vec{w} = 4\vec{a} - 2\vec{b}$

where $\vec{a}$ and $\vec{b}$ are not collinear, find the values of x and y for which $2\vec{u} - \vec{v} = \vec{w}$.

$2(x\vec{a} + 2y\vec{b}) - (-2y\vec{a} + 3y\vec{b}) = 4\vec{a} - 2\vec{b}$

$(2x+2y)\vec{a} + (4y-3y)\vec{b} = 4\vec{a} - 2\vec{b}$

$(2x+2y)\vec{a} + y\vec{b} = 4\vec{a} - 2\vec{b}$

$\therefore 2x+2y = 4$ ① $\qquad$ $y = -2$ ②

sub $y=-2$ into ① :

$2x + 2(-2) = 4$

$2x = 8$

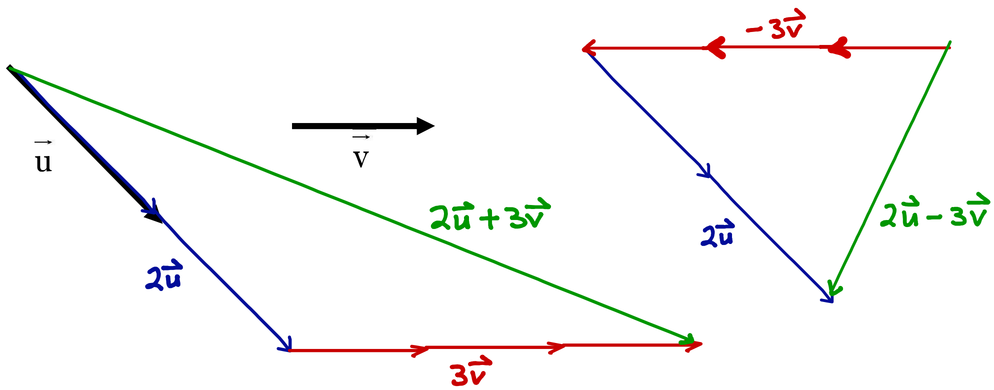
$x = 4$

$\therefore x = 4$ and $y = -2$

Warm-up:

1. For vectors $\vec{u}$ and $\vec{v}$ shown below, draw a diagram of

a) $2\vec{u}+3\vec{v}$ b) $2\vec{u}-3\vec{v}$



2. Name a single vector equal to each combination of vectors.

(a) $\overrightarrow{AB}+\overrightarrow{BC} = \overrightarrow{AC}$

(b) $\overrightarrow{AB}+\overrightarrow{BD} = \overrightarrow{AD}$

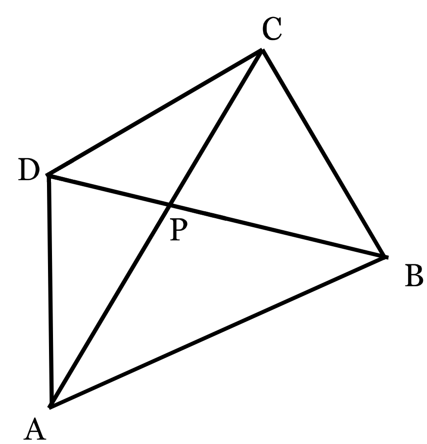
(c) $\overrightarrow{CD}+\overrightarrow{DA} = \overrightarrow{CA}$

(d) $\overrightarrow{BC}-\overrightarrow{DC} = \overrightarrow{BC}+\overrightarrow{CD} = \overrightarrow{BD}$

(e) $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD} = \overrightarrow{AD}$

(f) $\overrightarrow{DC}-\overrightarrow{BC}+\overrightarrow{BD} = \overrightarrow{DC}+\overrightarrow{CB}+\overrightarrow{BD} = \vec{0}$

(g) $\overrightarrow{AB}+\overrightarrow{BP}-\overrightarrow{CP}+\overrightarrow{CB} = \overrightarrow{AB}+\overrightarrow{BP}+\overrightarrow{PC}+\overrightarrow{CB} = \overrightarrow{AB}$



3. If $\vec{u} = a\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{v} = (b,-15,c)$ are collinear vectors, find

(a) c

(b) a relationship between a and b.

Let $\vec{u} = k\vec{v}$ since $\vec{u}$ and $\vec{v}$ are collinear

$\therefore a\hat{i}+5\hat{j}-3\hat{k} = k(b\hat{i}-15\hat{j}+c\hat{k})$, $k \in \mathbb{R}$

or $[a,5,-3] = k[b,-15,c]$

$\therefore 5 = -15k$
$\therefore k = -\frac{1}{3}$

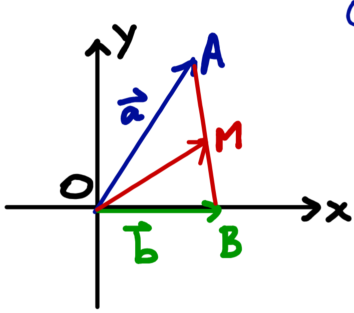
$\therefore -3 = kc$
$\therefore -3 = (-\frac{1}{3}c)$
$\boxed{c = 9}$

$\therefore a = kb$
$\therefore a = -\frac{1}{3}b$
or $\boxed{b = -3a}$

Position Vectors

A position vector is a vector with the additional property that it is fixed at its tail to the origin O. This is not a free vector, since O is a fixed point $\overrightarrow{OP}=\vec{p}$.

Example 1: In ΔAOB, $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$. Let M be the midpoint of $\overline{AB}$. Find the vector $\overrightarrow{OM}$ in terms of $\vec{a}$ and $\vec{b}$.



$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OA}+\overrightarrow{AM}\\ &= \overrightarrow{OA}+\tfrac{1}{2}\overrightarrow{AB}\\ &= \vec{a}+\tfrac{1}{2}(-\vec{a}+\vec{b})\\ &= \vec{a}-\frac{\vec{a}}{2}+\frac{\vec{b}}{2}\\ &= \frac{\vec{a}}{2}+\frac{\vec{b}}{2}\\ &= \tfrac{1}{2}(\vec{a}+\vec{b})\end{aligned}$$

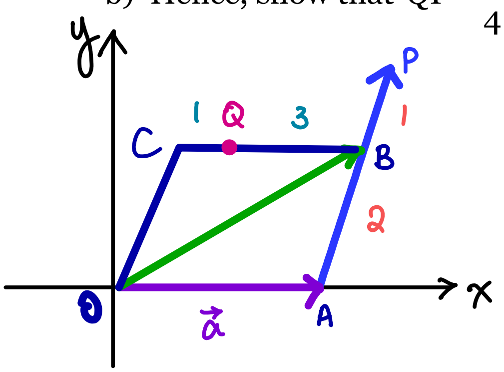
Note: $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$
$= -\overrightarrow{OA}+\overrightarrow{OB}$

Example 2: OABC is a parallelogram with $\overrightarrow{OA}=\vec{a}$ and $\overrightarrow{OB}=\vec{b}$. The point P lies on AB extended such that AB : BP = 2 : 1, and the point Q lies on CB such that CQ : QB = 1 : 3.

a) Express each Of these vectors terms of $\vec{a}$ and $\vec{b}$.

i) $\overrightarrow{AB}$ *ii)* $\overrightarrow{AP}$ *ii)* $\overrightarrow{OP}$ *iv)* $\overrightarrow{OQ}$

b) Hence, show that $\overrightarrow{QP}=\frac{1}{4}\vec{a}+\frac{1}{2}\vec{b}$



a) i) $\overrightarrow{AB}=\overrightarrow{AO}+\overrightarrow{OB}$
$= -\overrightarrow{OA}+\overrightarrow{OB}$
$= -\vec{a}+\vec{b}$
$= \vec{b}-\vec{a}$

ii) $\overrightarrow{AB}:\overrightarrow{BP}=2:1$
$\therefore \overrightarrow{AB}=\frac{2}{3}\overrightarrow{AP}$
$\overrightarrow{AP}=\frac{3}{2}\overrightarrow{AB}$
$=\frac{3}{2}(\vec{b}-\vec{a})$

iii) $\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}$
$=\vec{a}+\frac{3}{2}\vec{b}-\frac{3}{2}\vec{a}$
$=\frac{3}{2}\vec{b}-\frac{1}{2}\vec{a}$

iv) $\overrightarrow{OQ}=\overrightarrow{OB}+\overrightarrow{BQ}$
$=\overrightarrow{OB}+\frac{3}{4}\overrightarrow{BC}$
$=\overrightarrow{OB}-\frac{3}{4}\overrightarrow{CB}$
$=\overrightarrow{OB}-\frac{3}{4}\overrightarrow{OA}$
$=\vec{b}-\frac{3}{4}\vec{a}$

Note:
$\overrightarrow{CQ}:\overrightarrow{QB}=1:3$
$\frac{\overrightarrow{BQ}}{\overrightarrow{BC}}=\frac{3}{4}$
$\overrightarrow{BQ}=\frac{3}{4}\overrightarrow{BC}$

b) $\overrightarrow{QP}=\overrightarrow{QO}+\overrightarrow{OP}$
$=\overrightarrow{OP}-\overrightarrow{OQ}$
$=\frac{3}{2}\vec{b}-\frac{1}{2}\vec{a}-\vec{b}+\frac{3}{4}\vec{a}$
$=\frac{1}{4}\vec{a}+\frac{1}{2}\vec{b}$

Collinear points: points that lie on the same straight line

Example 3: The position vectors of the points, A, B and C are $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+2\hat{j}-\hat{k}$ and $6\hat{i}+11\hat{j}-7\hat{k}$, respectively, Show that A, B and C are collinear.

Let $\vec{a}=[2,-1,1]$, $\vec{b}=[3,2,-1]$ and $\vec{c}=[6,11,-7]$

$$\begin{aligned}\overrightarrow{AB}&=\overrightarrow{AO}+\overrightarrow{OB}\\&=\overrightarrow{OB}-\overrightarrow{OA}\\&=\vec{b}-\vec{a}\\&=[3,2,-1]-[2,-1,1]\\&=[1,3,-2]\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC}&=\overrightarrow{BO}+\overrightarrow{OC}\\&=\overrightarrow{OC}-\overrightarrow{OB}\\&=\vec{c}-\vec{b}\\&=[6,11,-7]-[3,2,-1]\\&=[3,9,-6]\\&=3[1,3,-2]\\&=3\overrightarrow{AB}\end{aligned}$$

Since $\overrightarrow{BC}$ is a scalar multiple of $\overrightarrow{AB}$; hence, $\overrightarrow{BC}$ and $\overrightarrow{AB}$ are collinear with B as a common point.

Example 4: The position vectors of a triangle ABC are $\overrightarrow{OA}=\begin{pmatrix}4\\5\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-3\\6\end{pmatrix}$, and $\overrightarrow{OC}=\begin{pmatrix}7\\-2\end{pmatrix}$.

a) Find $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$ and show that $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}=\vec{0}$.

b) Find $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$ and $|\overrightarrow{AB}+2\overrightarrow{BC}|$

a) $$\begin{aligned}\overrightarrow{AB}&=\overrightarrow{AO}+\overrightarrow{OB}\\&=-\overrightarrow{OA}+\overrightarrow{OB}\\&=-\begin{pmatrix}4\\5\end{pmatrix}+\begin{pmatrix}-3\\6\end{pmatrix}\\&=\begin{pmatrix}-7\\1\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC}&=\overrightarrow{BO}+\overrightarrow{OC}\\&=-\begin{pmatrix}-3\\6\end{pmatrix}+\begin{pmatrix}7\\-2\end{pmatrix}\\&=\begin{pmatrix}10\\-8\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\overrightarrow{CA}&=\overrightarrow{CO}+\overrightarrow{OA}\\&=-\begin{pmatrix}7\\-2\end{pmatrix}+\begin{pmatrix}4\\5\end{pmatrix}\\&=\begin{pmatrix}-3\\7\end{pmatrix}\end{aligned}$$

$$\begin{aligned}\therefore\ \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA}&=\begin{pmatrix}-7\\1\end{pmatrix}+\begin{pmatrix}10\\-8\end{pmatrix}+\begin{pmatrix}-3\\7\end{pmatrix}\\&=\begin{pmatrix}0\\0\end{pmatrix}\\&=\vec{0}\end{aligned}$$

b) $$\begin{aligned}|\overrightarrow{OA}|&=\sqrt{4^2+5^2}\\&=\sqrt{41}\end{aligned}$$

$$\begin{aligned}\overrightarrow{OB}&=\sqrt{(-3)^2+6^2}\\&=\sqrt{45}\end{aligned}$$

$$\begin{aligned}\therefore\ &|\overrightarrow{AB}+2\overrightarrow{BC}|\\&=\left|\begin{pmatrix}-7\\1\end{pmatrix}+2\begin{pmatrix}10\\-8\end{pmatrix}\right|\\&=\left|\begin{pmatrix}13\\-15\end{pmatrix}\right|\\&=\sqrt{13^2+(-15)^2}\\&=\sqrt{394}\end{aligned}$$

Practice

1. Relative to a fixed origin O, the points A, B, and C have position vectors (-2, 7, 4), (-4,1,8) and (6, -5, 0) respectively.
   (a) Find the position vector of the midpoint of AB.
   (b) Find the position vector of the point D on AC such that $AD : DC = 3:1$.
   Hint: Do not attempt to do this problem with distances. How would you split a line into 4 equal segments using midpoints?

2. Relative to a fixed origin o, the point A, B, and C have position vectors (6, -2, -4), (12, -7, -4), and (6, 1, -8) respectively.
   (a) Find the position vector of the point M, the midpoint of BC.
   (b) Show that O, A, and M are collinear points.

3. Given that $\vec{p}$ = (1, -2, 4), $\vec{q}$ = (-1, 2, 2), and $\vec{r}$ = (2, -4, -7). Find the value of *t* such that $\vec{p}+t\vec{q}$ is parallel to $\vec{r}$.

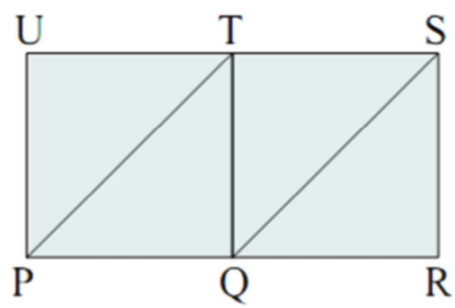
4. The diagram contains two squares. Express each difference as a single vector

   a) $\overrightarrow{SQ}-\overrightarrow{ST}$

   b) $\overrightarrow{QT}-\overrightarrow{QP}$

   c) $\overrightarrow{PR}-\overrightarrow{QS}$

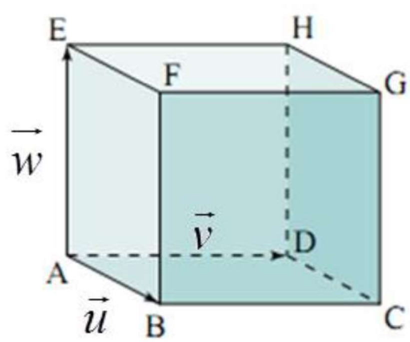
   d) $\overrightarrow{PT}-\overrightarrow{TS}$



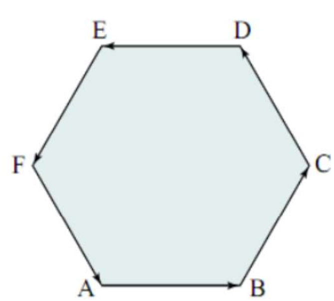
5. The diagram shows a cube, where
   $\overrightarrow{AB}=\vec{u}$, $\overrightarrow{AD}=\vec{v}$ and $\overrightarrow{AD}=\vec{v}$ .Determine a single vector equivalent to each of the following.

   a) $\vec{u}-\vec{v}+\vec{w}$

   b) $\vec{u}-\vec{v}-\vec{w}$



6. The diagram shows a regular hexagon.
   Prove that: $\overrightarrow{AB}-\overrightarrow{BC}+\overrightarrow{CD}-\overrightarrow{DE}+\overrightarrow{EF}-\overrightarrow{FA}=\vec{o}$



Practice

1. Relative to a fixed origin O, the points A, B, and C have position vectors (-2, 7, 4), (-4,1,8) and (6, -5, 0) respectively.
   (a) Find the position vector of the midpoint of AB.
   (b) Find the position vector of the point D on AC such that *AD* : *DC*= 3: 1.
   Hint: Do not attempt to do this problem with distances. How would you split a line into 4 equal segments using midpoints?

a)

$$M = \left(\frac{-2-4}{2}, \frac{7+1}{2}, \frac{4+8}{2}\right)$$

$$M = (-3,4,6)$$

$$\overrightarrow{OM} = [-3,4,6]$$

b)

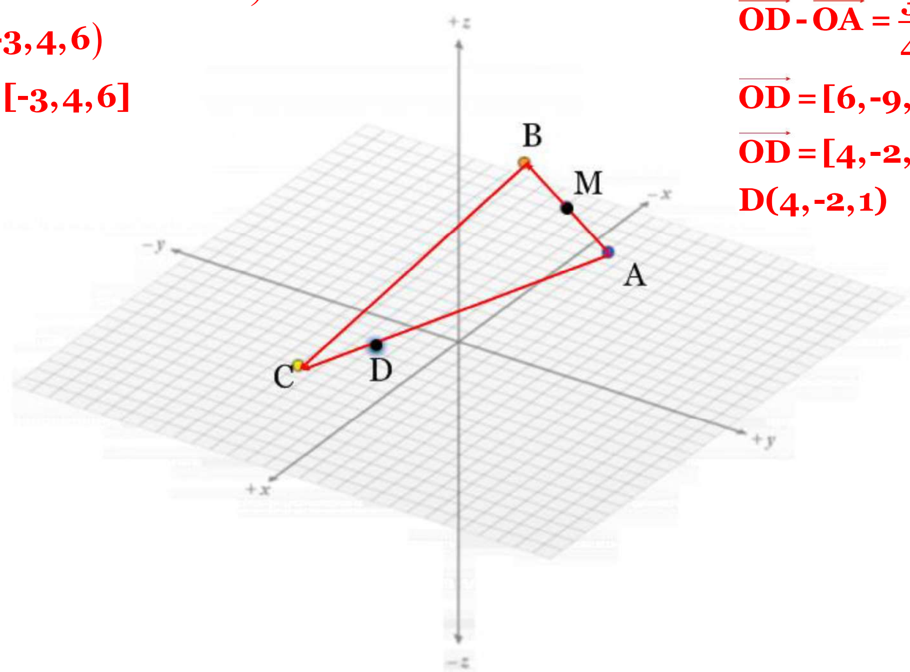
$$\overrightarrow{AD} = \frac{3}{4}\overrightarrow{AC}$$

$$\overrightarrow{OD} - \overrightarrow{OA} = \frac{3}{4}[8,-12,-4]$$

$$\overrightarrow{OD} = [6,-9,-3]+[-2,7,4]$$

$$\overrightarrow{OD} = [4,-2,1]$$

$$D(4,-2,1)$$



2. Relative to a fixed origin O, the point A, B, and C have position vectors (6, -2, -4), (12, -7, -4), and (6, 1, -8) respectively.
   (a) Find the position vector of the point M, the midpoint of BC.
   (b) Show that O, A, and M are collinear points.

a)

$$M = \left(\frac{12+6}{2}, \frac{-7+1}{2}, \frac{-4-8}{2}\right)$$

$$M = (9,-3,-6)$$

$$\overrightarrow{OM} = [9,-3,-6]$$

b)

$$\overrightarrow{OA} = k\overrightarrow{OM}$$

$$\overrightarrow{OA} = [6,-2,-4]$$

$$\overrightarrow{OM} = [9,-3,-6]$$

$$[6,-2,-4] = \frac{2}{3}[9,-3,-6]$$

3. Given that $\vec{p}$ = (1, -2, 4), $\vec{q}$ = (-1, 2, 2), and $\vec{r}$ = (2, -4, -7). Find the value of t such that $\vec{p}+t\vec{q}$ is parallel to $\vec{r}$.

$\vec{p}+t\vec{q}=[1,-2,4]+t[-1,2,2]$

$\vec{r}=[2,-4,-7]$

$\vec{p}+t\vec{q} \parallel \vec{r} \Leftrightarrow \vec{p}+t\vec{q}=k\vec{r}$

$[1-t,-2+2t,4+2t]=[2k,-4k,-7k]$

$1-t=2k \quad \uparrow t=1-2k$

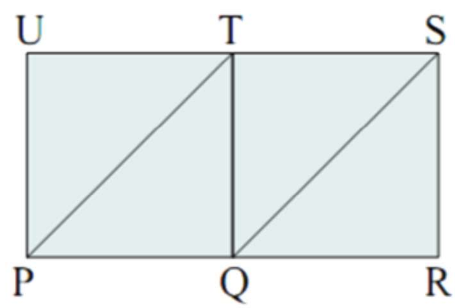
$-2+2t=-4k$

$4+2t=-7k \quad \uparrow 4+2(1-2k)=-7k$

$6=-3k$

$\boxed{k=-2}$ **and** $\boxed{t=5}$

4. The diagram contains two squares. Express each difference as a single vector



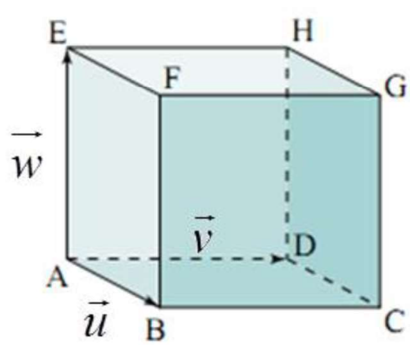
a) $\overrightarrow{SQ}-\overrightarrow{ST}=\overrightarrow{SQ}+\overrightarrow{TS}$
$=\overrightarrow{TQ}$

b) $\overrightarrow{QT}-\overrightarrow{QP}=\overrightarrow{PT}$

c) $\overrightarrow{PR}-\overrightarrow{QS}=\overrightarrow{PR}+\overrightarrow{SQ}$
$=\overrightarrow{PR}+\overrightarrow{TP}$
$=\overrightarrow{TR}$

d) $\overrightarrow{PT}-\overrightarrow{TS}=\overrightarrow{QS}+\overrightarrow{ST}$
$=\overrightarrow{QT}$

5. The diagram shows a cube, where $\overrightarrow{AB}=\vec{u}$,$\overrightarrow{AD}=\vec{v}$ and $\overrightarrow{AD}=\vec{v}$ .Determine a single vector equivalent to each of the following.



a) $\vec{u}-\vec{v}+\vec{w}=\overrightarrow{DF}$

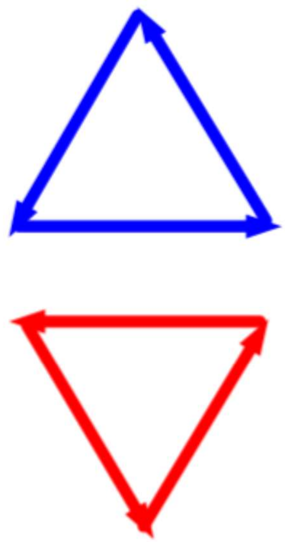
b) $\vec{u}-\vec{v}-\vec{w}=\overrightarrow{HB}$

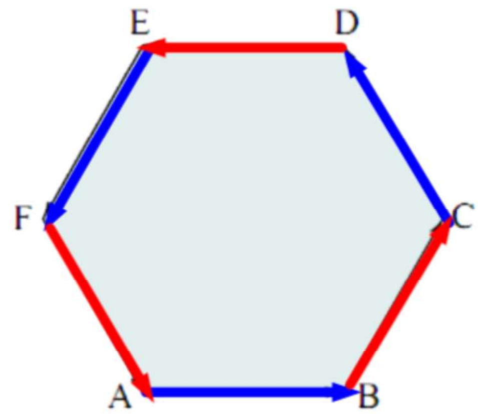
6. The diagram shows a regular hexagon.
   Prove that: $\overrightarrow{AB} - \overrightarrow{BC} + \overrightarrow{CD} - \overrightarrow{DE} + \overrightarrow{EF} - \overrightarrow{FA} = \vec{0}$

$$= \left(\overrightarrow{AB} + \overrightarrow{EF} + \overrightarrow{CD}\right) - \left(\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA}\right)$$

$$= \vec{0} - \vec{0}$$

$$= \vec{0}$$





3-5 Warm Up

1. Determine the values of r and s given that $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix}$ is parallel to $\vec{b} = \begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$.

2. Quadrilateral ORST has position vectors $\vec{r}$, $\vec{s}$ and $\vec{t}$. Point A is the midpoint of RS and point B divides ST such that SB:BT=2:5. Express each of these vectors in terms of $\vec{r}$, $\vec{s}$ and $\vec{t}$.

a) $\overrightarrow{RS}$ b) $\overrightarrow{ST}$ c) $\overrightarrow{OB}$ d) $\overrightarrow{AB}$

Solutions:

① $\vec{a} = k\vec{b}$

$\begin{pmatrix} 2 \\ -1 \\ r \end{pmatrix} = k\begin{pmatrix} s \\ 2 \\ -3 \end{pmatrix}$

$\therefore -1 = 2k$

$-\frac{1}{2} = k$

$2 = sk$

sub $k = -\frac{1}{2}$,

$2 = s\left(-\frac{1}{2}\right)$

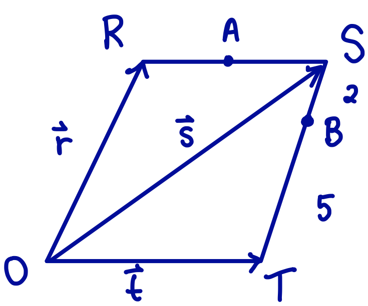
$-4 = s$

$r = -3k$, sub in $k = -\frac{1}{2}$,

$r = -3\left(-\frac{1}{2}\right)$

$r = \frac{3}{2}$

②



a) $\overrightarrow{RS} = -\overrightarrow{OR} + \overrightarrow{OS}$

$= \vec{s} - \vec{r}$

b) $\overrightarrow{ST} = -\overrightarrow{OS} + \overrightarrow{OT}$

$= \vec{t} - \vec{s}$

c) $\overrightarrow{OB} = \overrightarrow{OS} + \overrightarrow{SB}$

$= \vec{s} + \frac{2}{7}\overrightarrow{ST}$

$= \vec{s} + \frac{2}{7}(\vec{t} - \vec{s})$

$= \frac{5}{7}\vec{s} + \frac{2}{7}\vec{t}$

d) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OR} - \overrightarrow{RA} + \overrightarrow{OB}$

$= -\vec{r} - \frac{1}{2}\overrightarrow{RS} + \overrightarrow{OB}$

$= -\vec{r} - \frac{1}{2}(\vec{s} - \vec{r}) + \frac{5}{7}\vec{s} + \frac{2}{7}\vec{t}$

$= -\frac{1}{2}\vec{r} + \frac{3}{14}\vec{s} + \frac{2}{7}\vec{t}$

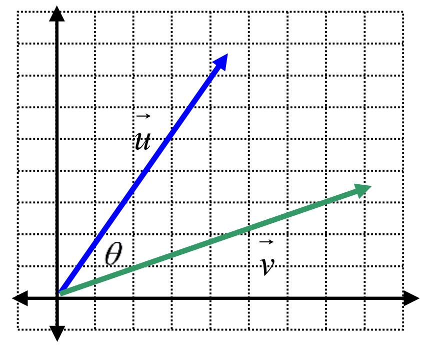
Dot Product of 2 Vectors – aka Scalar (Inner) Product

Dot Product :

- defined as: **(horizontal displacement of an object)**
- dot product involves two scalars
- result is a scalar ie) positive/negative/zero

$$\boxed{\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)}$$

Note: Vectors need to be tail to tail



angle value	$\cos(\theta)$ **value**	$\vec{u} \bullet \vec{v}$
$0^\circ \le \theta < 90^\circ$	positive	positive
$\theta = 90^\circ$	0	0
$90^\circ < \theta \le 180^\circ$	negative	negative

Example 1: Given vectors $\vec{u}$ and $\vec{v}$, where $|\vec{u}| = 10$ and $|\vec{v}| = 13$ and the angle between them is 150°, calculate $\vec{u} \bullet \vec{v}$.

$\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

$= (10)(13)\cos(150^\circ)$

$= -65\sqrt{3}$

Dot Product Properties

1) ***Commutative***: $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$

2) ***Distributive*** *over vector addition:* $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$

3) ***Associative*** *over scalar multiplication:* $m(\vec{u} \bullet \vec{v}) = (m\vec{u}) \bullet \vec{v} = \vec{u} \bullet (m\vec{v})$

$(m\vec{u}) \bullet (n\vec{v}) = mn(\vec{u} \bullet \vec{v})$

Example 2:Evaluate $\hat{j} \bullet \hat{j}$ and $\hat{i} \bullet \hat{j}$.

$\hat{j} \bullet \hat{j} = (1)(1)\cos(0^\circ)$

$= 1$

$\hat{i} \bullet \hat{j} = (1)(1)\cos(90^\circ)$

$= 0$

Example 3: What is the dot product of a vector $\vec{u}$ with itself? ($\theta = 0^\circ$).

$\vec{u} \bullet \vec{u} = |\vec{u}||\vec{u}|\cos(0^\circ)$

$= |\vec{u}|^2$

Example 4: If vectors $3\vec{p}+\vec{q}$ and $\vec{p}-3\vec{q}$ are perpendicular and $|\vec{p}| = 2|\vec{q}|$, determine the angle between the non-zero vectors $\vec{p}, \vec{q}$.

$$(3\vec{p}+\vec{q})\cdot(\vec{p}-3\vec{q})=0$$

$$3\vec{p}\cdot\vec{p}-9\vec{p}\cdot\vec{q}+\vec{p}\cdot\vec{q}-3\vec{q}\cdot\vec{q}=0$$

$$3|\vec{p}|^2-8|\vec{p}||\vec{q}|\cos(\theta)-3|\vec{q}|^2=0$$

$\because |\vec{p}|=2|\vec{q}| \rightarrow$ $3(2|\vec{q}|)^2-8(2|\vec{q}|)|\vec{q}|\cos(\theta)-3|\vec{q}|^2=0$

$$9|\vec{q}|^2-16|\vec{q}|^2\cos(\theta)=0$$

$$\cos(\theta)=\frac{9}{16}$$

$$\theta \doteq 55.8^\circ$$

How to Evaluate Dot Product of Algebraic Vectors

2-dimensions	**3-dimensions**
$\vec{u}=(u_1,u_2),\vec{v}=(v_1,v_2)$	$\vec{u}=(u_1,u_2,u_3),\vec{v}=(v_1,v_2,u_3)$
$\vec{u}\bullet\vec{v}=u_1v_1+u_2v_2$	$\vec{u}\bullet\vec{v}=u_1v_1+u_2v_2+u_3v_3$

Example 5. Given that vectors $\vec{u}=\begin{pmatrix}k+2\\5\end{pmatrix}$ and $\vec{v}=\begin{pmatrix}k+1\\-6\end{pmatrix}$ are perpendicular, solve for k.

$\because \vec{u}\perp\vec{v} \therefore \vec{u}\cdot\vec{v}=0$

$$\begin{pmatrix}k+2\\5\end{pmatrix}\cdot\begin{pmatrix}k+1\\-6\end{pmatrix}=0$$

$$(k+2)(k+1)+(5)(-6)=0$$

$$k^2+3k+2-30=0$$

$$k^2+3k-28=0$$

$$(k+7)(k-4)=0$$

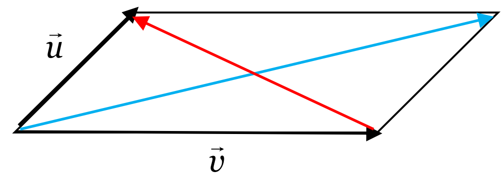
$$\therefore k=-7, k=4$$

Using dot product to find angle between two vectors

The formula $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ can be rearranged to make solving for θ simpler.

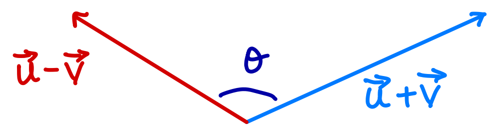
$$\boxed{\cos(\theta) = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|}}$$

Example 6: A parallelogram is bounded by vectors $\vec{u} = (1,2)$ and $\vec{v} = (3,-2)$. Find the angle between the diagonals of the parallelogram



$\vec{u} + \vec{v} = [1,2] + [3,-2]$
$= [4,0]$

$\vec{u} - \vec{v} = [1,2] - [3,-2]$
$= [-2,4]$



$$\cos(\theta) = \frac{(\vec{u}+\vec{v}) \cdot (\vec{u}-\vec{v})}{|\vec{u}+\vec{v}||\vec{u}-\vec{v}|}$$

$$= \frac{[4,0] \cdot [-2,4]}{\sqrt{4^2+0^2}\ \sqrt{(-2)^2+4^2}}$$

$$= \frac{-8}{4(2\sqrt{5})}$$

$$= -\frac{1}{\sqrt{5}}$$

$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

$\theta \doteq 117^\circ$

Example 7: The diagram shows a parallelogram ABCD. The coordinates of A, B, and D are A(1,2,3) ,B(6,4,4) and D(2,5,5).

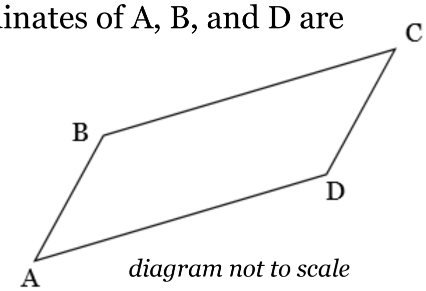


diagram not to scale

(a) Show that $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$, find $\overrightarrow{AD}$ and hence show that $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$.

(b) Find the coordinates of point C.

(c) Find $\overrightarrow{AB} \bullet \overrightarrow{AD}$ and hence find angle A.

a) $\vec{AB} = \begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$

$\vec{AD} = \begin{pmatrix} 2 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

$\vec{AC} = \vec{AD} + \vec{AB}$
$= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$
$= \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$

b) $\vec{AC} = \vec{AO} + \vec{OC}$
$\therefore \vec{OC} = \vec{AC} + \vec{OA}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$
$= \begin{pmatrix} 7 \\ 7 \\ 6 \end{pmatrix}$ $\therefore$ Pt C is (7,7,6).

c) $\vec{AB} \cdot \vec{AD} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$
$= 5(1) + 2(3) + 1(2)$
$= 13$

$$\cos(\theta) = \frac{(5,2,1)\cdot(1,3,2)}{\sqrt{5^2+2^2+1^2}\ \sqrt{1^2+3^2+2^2}}$$
$$= \frac{13}{\sqrt{30}\sqrt{14}}$$
$$= \frac{13}{2\sqrt{105}}$$

$\theta = \cos^{-1}\left(\frac{13}{2\sqrt{105}}\right)$

$\theta \doteq 50.6^\circ$

Practice

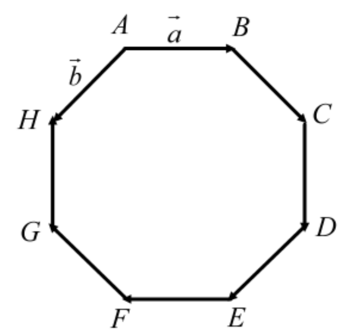
1. Given that $|\vec{a}| = 2, |\vec{b}| = 3$, and $\theta = 120^\circ$ expand and simplify $\left(3\vec{a}+4\vec{b}\right)\bullet\left(5\vec{a}+6\vec{b}\right)$.
2. The points A(-1,1), B(2,0), and C(1,-3) are vertices of a triangle.
   (a) Show that this triangle is a right triangle.
   (b) Calculate the area of triangle ABC.
   (c) Calculate the perimeter of triangle ABC.
   (d) Calculate the coordinates of the fourth vertex D that completes the rectangle of which A, B, and C are the other three vertices

3. If $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$, prove that the non-zero vectors $\vec{a}, \vec{b}$ are perpendicular. What could this look like?

4. Given the vectors $\vec{u} = [1,0,1]$, $\vec{v} = 2\hat{\imath} + \boldsymbol{m}\hat{\jmath} + 2\hat{k}$ find the value(s) of $\boldsymbol{m}$ if the angle between $\vec{u}$ and $\vec{v}$ is 45°.
5. Find the angle between the given vector and the axis.

   a) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ and negative x-axis

   b) $\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$ and positive x-axis
6. ABCDEFGH is a regular octagon with sides of unit length. (Recall interior angles are 135°). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AH} = \vec{b}$. Prove that $\overrightarrow{BC} = \vec{b} + \sqrt{2}\,\vec{a}$.



Practice

1. Given that $|\vec{a}| = 2, |\vec{b}| = 3$, and $\theta = 120^\circ$ expand and simplify $(3\vec{a} + 4\vec{b}) \bullet (5\vec{a} + 6\vec{b})$.

$$(3\vec{a}+4\vec{b})\bullet(5\vec{a}+6\vec{b}) = \mathbf{15}|\vec{\mathbf{a}}|^2 + \mathbf{18}\vec{\mathbf{a}}\bullet\vec{\mathbf{b}} + \mathbf{20}\vec{\mathbf{b}}\bullet\vec{\mathbf{a}} + \mathbf{24}|\vec{\mathbf{b}}|^2$$

$$= 15(2)^2 + 38|\vec{a}||\vec{b}|\cos(120^\circ) + 24(3)^3$$

$$= 60 + 38(2)(3)\left(\frac{-1}{2}\right) + 72$$

$$= 162$$

2. The points A(-1,1), B(2,0), and C(1,-3) are vertices of a triangle.
   (a) Show that this triangle is a right triangle.
   (b) Calculate the area of triangle ABC.
   (c) Calculate the perimeter of triangle ABC.
   (d) Calculate the coordinates of the fourth vertex D that completes the rectangle of which A, B, and C are the other three vertices

a)

$$\overrightarrow{BA} \perp \overrightarrow{BC} \Leftrightarrow \overrightarrow{BA} \bullet \overrightarrow{BC} = 0$$

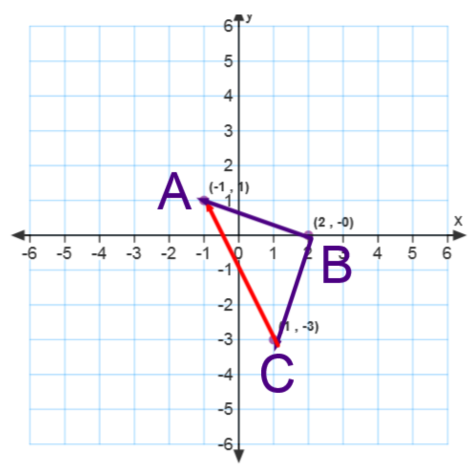
$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = [-1,1] - [2,0] = [-3,1]$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = [1,-3] - [2,0] = [-1,-3]$$

$$\overrightarrow{BA} \bullet \overrightarrow{BC} = [-3,1] \bullet [-1,-3]$$

$$= 3 - 3$$

$$= 0$$

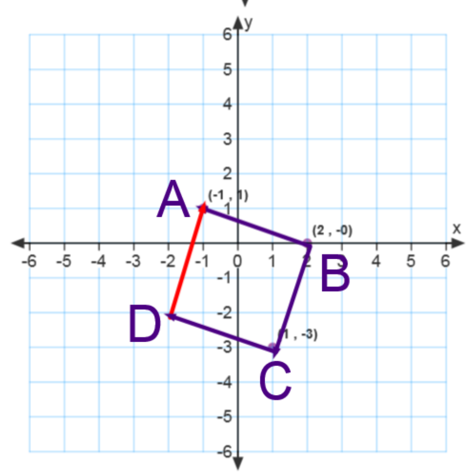


b)

$$A_{\Delta ABC} = \frac{1}{2}|\overrightarrow{BA}||\overrightarrow{BC}|$$

$$= \frac{1}{2}\left(\sqrt{9+1}\right)\left(\sqrt{1+9}\right)$$

$$= 5 \text{ units}^2$$



c)

$$P_{\Delta ABC} = |\overrightarrow{BA}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|$$

$$= \sqrt{10} + \sqrt{10} + \sqrt{20}$$

$$= 2\sqrt{5}\left(\sqrt{2}+1\right) \text{unit}$$

d)

$$\overrightarrow{BA} = \overrightarrow{CD}$$

$$\overrightarrow{BA} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$\overrightarrow{BA} + \overrightarrow{OC} = \overrightarrow{OD}$$

$$[-3,1] + [1,-3] = \overrightarrow{OD}$$

$$[-2,-2] = \overrightarrow{OD}$$

$$\therefore D(-2,-2)$$

3. If $\left|\vec{a}+\vec{b}\right| = \left|\vec{a}-\vec{b}\right|$, prove that the non-zero vectors $\vec{a}, \vec{b}$ are perpendicular. What could this look like?

$$\left|\vec{a}+\vec{b}\right|^2 = \left|\vec{a}-\vec{b}\right|^2$$

$$\left|\vec{a}+\vec{b}\right|^2 = \left(\vec{a}+\vec{b}\right)\cdot\left(\vec{a}+\vec{b}\right)$$

$$= \vec{a}\bullet\vec{a}+2\vec{a}\bullet\vec{b}+\vec{b}\bullet\vec{b}$$

$$= \left|\vec{a}\right|^2+2\vec{a}\bullet\vec{b}+\left|\vec{b}\right|^2$$

$$\left|\vec{a}-\vec{b}\right|^2 = \left(\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)$$

$$= \vec{a}\bullet\vec{a}-2\vec{a}\bullet\vec{b}+\vec{b}\bullet\vec{b}$$

$$= \left|\vec{a}\right|^2-2\vec{a}\bullet\vec{b}+\left|\vec{b}\right|^2$$

$$\left|\vec{a}\right|^2+\left|\vec{b}\right|^2+2\vec{a}\bullet\vec{b} = \left|\vec{a}\right|^2+\left|\vec{b}\right|^2-2\vec{a}\bullet\vec{b}$$

$$4\vec{a}\bullet\vec{b} = 0 \rightarrow \vec{a}\perp\vec{b}$$

4. Given the vectors $\vec{u} = [1,0,1]$, $\vec{v} = 2\hat{\imath} + \boldsymbol{m}\hat{\jmath} + 2\hat{k}$ find the value(s) of $\boldsymbol{m}$ if the angle between $\vec{u}$ and $\vec{v}$ is 45^{o} .

$$\cos(45^{\circ}) = \frac{\vec{u}\bullet\vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}$$

$$\frac{\sqrt{2}}{2} = \frac{4}{\left(\sqrt{1^2+0^2+1^2}\right)\left(\sqrt{2^2+m^2+2^2}\right)}$$

$$\sqrt{8+m^2} = 4$$

$$8+m^2 = 16$$

$$m^2 = 8$$

$$m = \pm 2\sqrt{2}$$

5. Find the angle between the given vector and the axis.

a) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ and negative x-axis

$$\cos(\theta) = \frac{[7,-3]\bullet[-1,0]}{\sqrt{7^2+(-3)^2}}$$

$$= \frac{-7}{\sqrt{58}}$$

$$\theta = 156.80^{o}$$

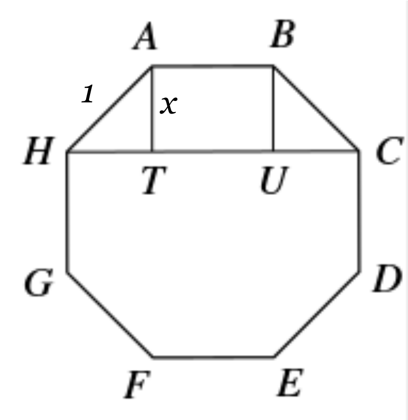
b) $\begin{pmatrix} 4 \\ 5 \\ 7 \end{pmatrix}$ and positive x-axis

$$\cos(\theta) = \frac{[4,5,7]\bullet[1,0,0]}{\sqrt{4^2+5^2+7^2}}$$

$$= \frac{4}{\sqrt{90}}$$

$$\theta = 87.45^{o}$$

6. ABCDEFGH is a regular octagon with sides of unit length. (Recall interior angles are 135°). Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{AH} = \vec{b}$. Prove that $\overrightarrow{BC} = \vec{b} + \sqrt{2}\ \vec{a}$.



$$\mathbf{AT} = \mathbf{TH} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$$

$$\mathbf{HC} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} + \mathbf{1} + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} = \mathbf{1} + \sqrt{\mathbf{2}}$$

$$\overrightarrow{\mathbf{HC}} \parallel \overrightarrow{\mathbf{AB}} \Rightarrow \overrightarrow{\mathbf{HC}} = \left(\mathbf{1} + \sqrt{\mathbf{2}}\right)\vec{\mathbf{a}}$$

$$\overrightarrow{\mathbf{BC}} = \overrightarrow{\mathbf{BA}} + \overrightarrow{\mathbf{AH}} + \overrightarrow{\mathbf{HC}} = -\vec{\mathbf{a}} + \mathbf{b} + (\mathbf{1} + \sqrt{\mathbf{2}})\vec{\mathbf{a}} = \vec{\mathbf{b}} + \sqrt{\mathbf{2}}\,\vec{\mathbf{a}}$$

$$\overrightarrow{\mathbf{FG}} = -\overrightarrow{\mathbf{BC}} = -\vec{\mathbf{b}} - \sqrt{\mathbf{2}}\,\vec{\mathbf{a}}$$

x

3-6 Warm Up

1. Find the dot (scalar) product of $\vec{u}$ and $\vec{v}$ for the following:
   a) $|\vec{u}| = 5, |\vec{v}| = 8, \ \theta = 13°$ b) $\vec{u} = (5, 6), \vec{v} = (-2, 3)$

2. Find the angle between the vectors:

   a) $\vec{u} = (-6, 1)$ and $\vec{v} = (-5, 3)$

   b) $3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $4\hat{\imath} - 5\hat{\jmath} - 3\hat{k}$

3. Find the value of c so that the vectors $\hat{\imath} + \hat{\jmath} + \hat{k}$ and $c^2\hat{\imath} - 2c\hat{\jmath} + \hat{k}$ are perpendicular.

Solutions:

① a) $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$

$= 5(8)\cos 13°$

$\doteq 39.0$

b) $\vec{u} \cdot \vec{v} = (5,6) \cdot (-2,3)$

$= -10 + 18$

$= 8$

② a) $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$

$= \dfrac{(-6,1) \cdot (-5,3)}{\sqrt{(-6)^2 + (1)^2}\sqrt{(-5)^2 + (3)^2}}$

$= \dfrac{(-6)(-5) + (1)(3)}{\sqrt{37}\sqrt{34}}$

$\theta = \cos^{-1}\left(\dfrac{33}{\sqrt{1258}}\right)$

$\theta \doteq 21.5°$ or 0.375 rad

b) $\cos\theta = \dfrac{3(4) + 4(-5) + (5)(-3)}{\sqrt{3^2 + 4^2 + 5^2}\sqrt{4^2 + (-5)^2 + (-3)^2}}$

$= \dfrac{-23}{\sqrt{50}\sqrt{50}}$

$= \dfrac{-23}{50}$

$\theta \doteq 117°$ or 2.05 rad

③ $(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (c^2\hat{\imath} - 2c\hat{\jmath} + \hat{k}) = 0$

$c^2 - 2c + 1 = 0$

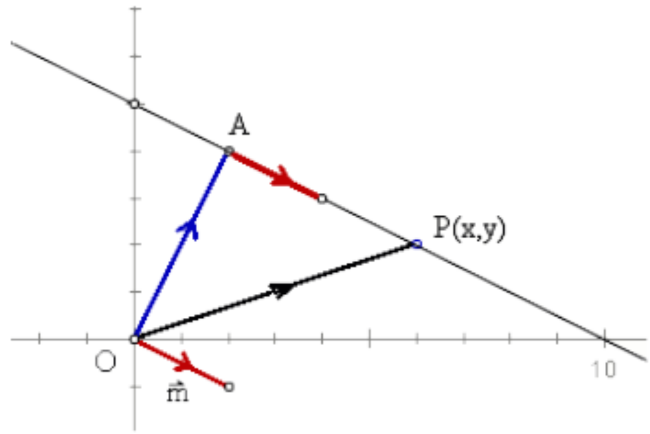
$(c-1)^2 = 0$

$\therefore c = 1$

Equations of Lines in R^2

In R^2 vectors can be used to define a line .Two new forms of the equation of the line are the **Vector Equation of a Line** and **Parametric Form of the Equation of a Line**. We start by defining the former.

Vectors can be used to locate points on a line as shown in the diagram at right. If A is a given point on the line and $\vec{m}$ is a vector parallel to the line, $\overrightarrow{OP} = \overrightarrow{OA} + t\vec{m}$ can be used to locate any point P(x, y) on the line.



This equation is called the **Vector Equation** of the line.

$$\overrightarrow{OP} = \overrightarrow{OA} + t\vec{m}$$

$\overrightarrow{OA}$ is called a **Position Vector**

$\vec{m}$ is called a **Direction Vector**

t is called a **Parameter** (any real number)

Vector Equation of Lines in R^2

Another way to write this equation using variables is $\vec{r} = \vec{r}_o + t\vec{m}$. By substituting $\vec{r} = (x, y)$, $\vec{r}_o = (x_o, y_o)$ and $\vec{m} = (m_1, m_2)$ into this equation we get another form of the vector equation.

The Vector Equation of a Line in R^2

$\vec{r} = \vec{r}_0 + t\vec{m}$ OR $(x, y) = (x_0, y_0) + t(m_1, m_2)$

or $\vec{r} = (x_0 + tm_1)\hat{i} + (y_0 + tm_2)\hat{j}, t \in \mathbb{R}$

or $\binom{x}{y} = \binom{x_0}{y_0} + t\binom{m_1}{m_2}, t \in \mathbb{R}$

where

- $t \in R$ is a parameter
- $\vec{r} = (x, y)$ is a position vector to any unknown point on the line
- $\vec{r}_0 = (x_0, y_0)$ is a position vector to any known point on the line
- $\vec{m} = (m_1, m_2)$ is a direction vector parallel to the line → any scalar multiple

Example 1:

a) Write a vector equation of a line passing through the points A(1, 4) and B(3, 1).

$\vec{m} = \overrightarrow{AB}$
$= [3,1] - [1,4]$
$= [2,-3]$

$\vec{r} = [1,4] + t[2,-3], t \in \mathbb{R}$

or $\vec{r} = [3,1] + t[2,-3], t \in \mathbb{R}$

b) Determine two more points on the line.

Choose any value of t:

If $t = 2$, $\vec{r} = [1,4] + 2[2,-3]$
$= [5,-2]$

If $t = -1$, $\vec{r} = [1,4] + (-1)[2,-3]$
$= [-1,7]$

$\therefore$ The two points are $(5,-2)$ and $(-1,7)$.

c) Determine if the point (2, 3) is on this line.

$\leftarrow$ t-value will be the same for x and y

If (2,3) lies on the line: $[2,3]=[1,4]+t[2,-3]$

$\therefore 2=1+2t$ and $3=4-3t$

$1=2t$ $\quad$ $-1=-3t$

$\frac{1}{2}=t$ $\quad$ $\frac{1}{3}=t$

Since the t-values are not the same
$\therefore$ The point (2,3) does not lie on the line.

NOTE: Vector equations are **NOT** unique!

The vector equation can be separated into two parts, one for each variable. These are called **parametric equations** of a line.

The Parametric form of the Equation of a Line in R²
For a line with equation $(x,y)=(x_0,y_0)+t(m_1,m_2)$, the parametric equations are $x = x_0 + tm_1$ $y = y_0 + tm_2$ where $t \in \mathrm{R}$ (the parameter)

Example 2: Rewrite your vector equation from Example 1(a) in parametric form.

From 1a) $[x,y]=[1,4]+t[2,-3], t\in\mathbb{R}$

Parametric form: $x=1+2t$
$y=4-3t$, $t\in\mathbb{R}$

NOTE: Again, like vector equations, parametric equations are not unique as we can use the coordinates of any point on the line and any scalar multiple of the direction vector.

Example 3: A line L_1 is defined by $x = 3 + t$ and $y = -5 + 2t$.

a) Find the coordinates of two points on this line.

Choose any value of t:

If $t=0$, $x=3$
$y=-5$

If $t=1$, $x=3+(1)$
$=4$
$y=-5+2(1)$
$=-3$

$\therefore$ Two points are (3,-5) and (4,-3)

b) Find the y-intercept of the line.

Sub x=0 to find parameter, t:

$0=3+t$
$-3=t$

sub $t=-3$ into y: $y=-5+2(-3)$
$=-11$

$\therefore$ y-int is -11.

$x = 3 + t$
$y = -5 + 2t$

c) Write the vector equation for L_1.

$[x, y] = [3, -5] + t[1, 2]$

or $[x, y] = [0, -11] + t[1, 2]$

d) Determine if L_1 is parallel to L_2: $x = 1 + 2t$, $y = -9 + 4t$.

L_1: $\vec{m}_1 = [1, 2]$

L_2: $\vec{m}_2 = [2, 4]$

Since $\vec{m}_2 = 2\vec{m}_1$

$\therefore$ L_1 and L_2 are parallel.

Extension: are L_1 and L_2 coincident?

Check if point $(3, -5)$ lies on L_2:

$3 = 1 + 2t$
$2 = 2t$
$1 = t$

$-5 = -9 + 4t$
$4 = 4t$
$1 = t$

$\because$ t is unique $\therefore$ point $(3, -5)$ lies on L_2.
$\therefore$ L_1 and L_2 are coincident.

Symmetric Equation of a line.

The Symmetric form of the Equation of a Line in R²

For a line with equation $(x, y) = (x_0, y_0) + t(m_1, m_2)$, the symmetric equation is

$$\frac{x - x_0}{m_1} = \frac{y - y_0}{m_2}, m_1, m_2 \neq 0$$

Note: m_1 and m_2 are called direction numbers

What happens if m₁ or m₂ is zero?

Let suppose $m_2 = 0$. In this case t will not exist in the parametric equation for y and so we will only solve the parametric equations for x for t. We then set those equal and acknowledge the parametric equation for y as follows,

$$\frac{x - x_0}{m_1}, y = y_0$$

Example 4: Write all three forms of the equation of the line that passes through the points A (2,-1) and B(4,-1).

$\vec{m} = \vec{AB}$
$= [2, 0]$

Vector equation:

$\vec{r} = [4, -1] + t[2, 0], t \in \mathbb{R}$

Parametric equations:

$x = 4 + 2t$
$y = -1$
$, t \in \mathbb{R}$

Symmetric equations:

$\frac{x-4}{2}, y = -1$

$Ax + By + C = 0$

Example 5: Consider the line with Cartesian equation $4x + 5y + 20 = 0$.

a) Determine its slope. How does the slope compare to the Cartesian equation?

$4x+5y+20=0$
$5y=-4x-20$
$y=-\frac{4}{5}x-4$

$m = \frac{-4}{5}$ ← Δy, ← Δx

↳ $m=\frac{-A}{B}$

b) Determine a vector equation of this line. How does the direction vector relate to the slope?

1) Determine 2 points to get $\vec{m}$:

x-int: $A(-5,0)$
y-int: $B(0,-4)$
$\vec{m}=[5,-4]$ → $[\Delta x, \Delta y]$

2) Vector equation:

$\vec{r}=[-5,0]+t[5,-4], t\in\mathbb{R}$

Note: $\vec{m}=[B,-A]$ → slope $=\frac{m_2}{m_1}$

c) Determine a position vector that is perpendicular to the line (e.g. a normal vector). How does the normal vector compare to the Cartesian equation?

Let the normal vector be $\vec{n}=[n_1,n_2]$; $\vec{m}=[5,-4]$

direction vector and normal vector are perpendicular ∴ dot product is zero.

$\vec{m}\cdot\vec{n}=0$
$[5,-4]\cdot[n_1,n_2]=0$
$5n_1-4n_2=0$

∴ $\vec{n}=[4,5]$ (or $\vec{n}=[8,10]$, etc. but in a Cartesian equation, we reduce to lowest terms)

* choose a value for n_1 and solve for n_2

In Cartesian equation: $\vec{n}=\frac{B}{A}$

NOTE: For a line with equation $Ax + By + C = 0$,

- the slope of the line is $m=\frac{-A}{B}$ and a direction vector $\vec{m}$ = $[B,-A]$.
- the normal vector is $\vec{n}$ = $[A,B]$.

You Try!

Determine equivalent vector, parametric, symmetric and Cartesian equations of the line $y=\frac{3}{4}x+2$.

$4y=3x+8$
$3x-4y+8=0$

∴ Cartesian equation is $3x-4y+8=0$

Vector equation:
$m=\frac{3}{4}$ ∴ $\vec{m}=[4,3]$
point $(0,2)$
∴ $\vec{r}=[0,2]+t[4,3], t\in\mathbb{R}$

Symmetric equations:
$\frac{x}{4}=\frac{y-2}{3}$

Parametric Equations:
$x=0+4t$
$y=2+3t$

Equations of Lines in R^3

As in R^2, a direction vector and a position vector to a known point on a line are all that are needed to define a line in R^3.

The Vector Equation of a Line in R^3

$\vec{r} = \vec{r}_0 + t\vec{m}$ OR $(x,y,z)=(x_0,y_0,z_0)+t(m_1,m_2,m_3)$

where

- $t \in \mathrm{R}$ is a parameter
- $\vec{r} = (x,y,z)$ is a position vector to any unknown point on the line
- $\vec{r}_0 = (x_0,y_0,z_0)$ is a position vector to any known point on the line
- $\vec{m} = (m_1,m_2,m_3)$ a direction vector parallel to the line ← or any scalar multiple

The Parametric form of the Equation of a Line in R^3

For a line with equation $(x,y,z)=(x_0,y_0,z_0)+t(m_1,m_2,m_3)$, the parametric equations are

$x = x_0 + tm_1$

$y = y_0 + tm_2$

$z = z_0 + tm_3$ where $t \in \mathrm{R}$

Overall, the various new forms of lines in ***R***^{2} can be extended to lines in ***R***^{3}.

Comparison of equations of a line in *R*^{2} and *R*^{3}

	Equation of a line in ***R***^{2}	Equation of a line in ***R***^{3}
Scalar	$Ax + By + C = 0$	
Vector	$(x,y)=(x_o,y_o)+t(m_1,m_2)$	$(x,y,z)=(x_o,y_o,z_o)+t(m_1,m_2,m_3)$
Parametric	$x = x_o + tm_1$ $y = y_o + tm_2$, $t \in \mathbb{R}$	$x = x_o + tm_1$ $y = y_o + tm_2$ $z = z_o + tm_3$, $t \in \mathbb{R}$
Symmetric	$\dfrac{x - x_o}{m_1} = \dfrac{y - y_o}{m_2}$ where $m_1, m_2 \neq 0$	$\dfrac{x - x_o}{m_1} = \dfrac{y - y_o}{m_2} = \dfrac{z - z_o}{m_3}$ where $m_1, m_2, m_3 \neq 0$

Example 6: A line passes through points A(2, -2, 5) and B(0, 6, -5).

a) Write a vector equation for the line.

b) Write parametric equations for the line.

c) Write symmetric equations for the line.

d) Determine if the point C(0, -10, 9) lies on the line.

a) $\vec{m} = \vec{AB}$
$= [0,6,-5] - [2,-2,5]$
$= [-2,8,-10]$

$\vec{r} = [2,-2,5] + t[-2,8,-10], t \in \mathbb{R}$

or $\vec{r} = [0,6,-5] + t[-1,4,-5], t \in \mathbb{R}$

b) $x = 2 - 2t$
$y = -2 + 8t$
$z = 5 - 10t \quad , t \in \mathbb{R}$

c) $\frac{x-2}{-2} = \frac{y+2}{8} = \frac{z-5}{-10}$

d) sub C(0,-10,9) into parametric equations and check t-values:

$0 = 2 - 2t$	$-10 = -2 + 8t$	$9 = 5 - 10t$
$-2 = -2t$	$-8 = 8t$	$4 = -10t$
$1 = t$	$-1 = t$	$-\frac{2}{5} = t$

$\therefore$ t-values are not the same, the point does not lie on the line.

Thinking Question: Why can't a normal vector and a point define a line in R3?

In $\mathbb{R}^3$, there are an infinite number of normals to a line that can pass through a point. Therefore, a normal vector and a point cannot define a unique line.

Practice

1. Determine if the following points are on the line ℓ:[−4,3]+t[3,2].

a) (−1,5) b) (−16,−5)

2. For the line defined by $\ell:\begin{cases} x=-3-t \\ y=2+2t \end{cases}$, state the coordinates of

a) the y-intercept
b) the x-intercept
c) the point where x=12
d) the point where y=38

3. Rewrite each of the equations below into the specified form.

a) ℓ:[7,2]+t[3,−2] into parametric form
b) $\ell:\begin{cases} x=32-3t \\ y=26+4t \end{cases}$ into vector form

4. Find the equation of the line and write in the specified form:

a) the line parallel to $\vec{m}=[2,3]$ that hits the point (1,4), in parametric form.
b) the line that passes through the points (2,4) and (5,13), in vector form.
c) the vertical line through (4,−2), in parametric form.
d) the line with the same x-intercept as ℓ_1:[3,6]+t[1,−2], and the same y-intercept as ℓ_2:[8,4]+s[−1,3], in vector form.

5. Given the line ℓ:[7,3,1]+t[−1,3,1], determine if the following lines are parallel, perpendicular, or coincident to it.

a) ℓ_2:[2,−3,4]+t[5,1,2]
b) ℓ_3: x=1+t , y=21−3t , z=7−t
c) ℓ_4:[5,3,2]+t[−2,6,2]
d) ℓ_5:[3,7,−2]+t[4,6,1]

6. If the points (4,2,7),(6,19,−4), and (80,b,c) lie on the same straight line, find the values of b and c.

7. Determine the angle between each pair of lines:

a) ℓ_1:[4,5,−2]+t[3,−1,−1] ℓ_2:[4,5,−2)+s[−2,−3,2]
b) $\ell_1:\frac{x-5}{3}=\frac{y+2}{5}=z-2$ $\ell_2:\frac{x-5}{8}=y+2=\frac{2-z}{3}$

8. Find, in parametric form, the equation of a line perpendicular to both ℓ_1:[3,7,−2]+t[3,−1,−1] and ℓ_2:[8,−3,−3]+t[−2,−3,2] that passes through (5,0,0).

9. Find, if possible, the value(s) of k such that the lines ℓ_1:[9,3,2]+t[3,k,−15] and ℓ_2:[−5,4,−2]+t[10,12,50] are:

a) parallel b) perpendicular

10. Point P_1 lies on the line ℓ_1:[4,4,−3]+t[2,1,−1],t∈ℝ, and point P_2 lies on the line ℓ_2:[−2,−7,2]+s[3,2,−3]. If the vector $\overrightarrow{P_1P_2}$ is perpendicular to both ℓ_1 and ℓ_2, determine the coordinates of P_1 and P_2.

Practice-**Solution**

1. Determine if the following points are on the line ℓ:[−4,3]+t[3,2].
a) (−1,5) **on ℓ**
b) (−16,−5) **on ℓ**

2. For the line defined by $\ell:\begin{cases} x=-3-t \\ y=2+2t \end{cases}$, state the coordinates of
a) the y-intercept **(0,−4)**
b) the x-intercept **(−2,0)**
c) the point where x=12 **(12,−28)**
d) the point where y=38 **(−21,38)**

3. Rewrite each of the equations below into the specified form.
a) ℓ:[7,2]+t[3,−2] into parametric form $\mathbf{l:}\begin{cases} \mathbf{x=7+3t} \\ \mathbf{y=2-2t} \end{cases}$
b) $\ell:\begin{cases} x=32-3t \\ y=26+4t \end{cases}$ into vector form **ℓ:[x,y,z]=[32,26]+t[-3,4]**

4. Find the equation of the line and write in the specified form:
a) the line parallel to $\vec{m}=[2,3]$ that hits the point (1,4), in parametric form.

$$\mathbf{x=1+2t}$$
$$\mathbf{y=4+3t \ \ ,t\in R}$$

b) the line that passes through the points (2,4) and (5,13), in vector form.

$$\vec{\mathbf{m}}=\overrightarrow{\mathbf{AB}}=[5,13]-[2,4]=[3,9]=3[1,3]$$
$$\vec{\mathbf{r}}=[2,4]+t[1,3] \ \ ,t\in R$$

c) the vertical line through (4,−2), in parametric form.

In two dimensional space a vertical line is parallel to y-axis , Therefore the direction vector of such a line is collinear with $\hat{\mathbf{j}}=[0,1]$.

$$\mathbf{x=4}$$
$$\mathbf{y=-2+t \ \ ,t\in R}$$

d) the line with the same x-intercept as ℓ_1:[3,6]+t[1,−2], and the same y-intercept as ℓ_2:[8,4]+s[−1,3], in vector form.

To find the x-intercept of $\ell_1:[3,6]+t[1,-2]$, we need to set $y=0$;i.e.
$6-2t=0$
$t=3$

sub. $t=3$ into x-component: we get $x=3+t$

$x=3+3$
$x=6$

$\therefore$ x-int:A $(6,0)$

To find the y-intercept of $\ell_2:[8,4]+s[-1,3]$ we need to set $x=0$;i.e.
$8-s=0$
$s=8$

sub. $s=8$ into y-component: we get $y=4+3s$

$y=4+3(8)$
$y=28$

$\therefore$ y-int:B $(0,28)$

$$\vec{m}=\overrightarrow{AB}=[0,28]-[6,0]=[-6,28]=-2[3,-14]$$

$$\therefore \vec{r}=[6,0]+t[3,-14] \quad ,t\in R$$

5. Given the line $\ell:[7,3,1]+t[-1,3,1]$, determine if the following lines are parallel, perpendicular, or coincident to it.

a) $\ell_2:[2,-3,4]+t[5,1,2]$ Perpendicular

b) ℓ_3: $x=1+t$, $y=21-3t$, $z=7-t$ Coincident

c) $\ell_4:[5,3,2]+t[-2,6,2]$ Parallel

d) $\ell_5:[3,7,-2]+t[4,6,1]$ None of these

6. If the points A (4,2,7),B (6,19,−4), and C(80,b,c) lie on the same straight line, find the values of b and c.

Three points A,B and C lie on the same straight line, iff they are collinear.
This means : $\overrightarrow{AC} = k\overrightarrow{AB}$

$$\overrightarrow{AB} = [6,19,-4]-[4,2,7]=[2,17,-11]$$
$$\overrightarrow{AC} = [6,19,-4]-[80,b,c]=[-74,19-b,-4-c]$$
$$\overrightarrow{AC} = k\overrightarrow{AB} \text{ iff } -74 = 2k \quad (1)$$
$$19-b=17k \quad (2)$$
$$-4-c=-11k \quad (3)$$

from (1) we get: $k=-37$
sub. $k=-37$ **into (2) & (3) we get:**

$$19-b=17(-37) \quad \& \quad -4-c=-11(-37)$$
$$\boxed{b=648} \qquad \boxed{c=-411}$$

7. Determine the angle between each pair of lines:
a) ℓ₁:[4,5,−2]+t[3,−1,−1] ℓ₂:[4,5,−2)+s[−2,−3,2]

$$\cos(\theta)=\frac{\overrightarrow{m_1}\bullet\overrightarrow{m_2}}{|\overrightarrow{m_1}||\overrightarrow{m_2}|}$$
$$=\frac{[3,-1,-1]\bullet[-2,-3,2]}{\left(\sqrt{3^2+1^2+1^2}\right)\left(\sqrt{2^2+3^2+2^2}\right)}$$
$$=\frac{-5}{\left(\sqrt{11}\right)\left(\sqrt{17}\right)}$$
$$=-0.3656$$
$$\theta=\cos^{-1}(-0.3656)$$
$$\theta \doteq 111.4^\circ \text{ or } 68.6^\circ$$

b) $\ell_1: \frac{x-5}{3}=\frac{y+2}{5}=z-2 \qquad \ell_2: \frac{x-5}{8}=y+2=\frac{2-z}{3}$ **59.3° or 120.7°**

The steps are the same as above

8. Find, in parametric form, the equation of a line perpendicular to both ℓ_1:[3,7,−2]+t[3,−1,−1] and ℓ_2:[8,−3,−3]+t[−2,−3,2] that passes through (5,0,0).

Let $\vec{m} = [x, y, z]$ represent the direction vector of the line perpendicular to both l_1 and l_2,

$\vec{m} \bullet [3,-1,-1] = 0$ **and** $\vec{m} \bullet [-2,-3,2] = 0$

$3x - y - z = 0 \qquad (1)$

$-2x - 3y + 2z = 0 \qquad (2)$

$(2) + 2\times(1)\ :\ 4x - 5y = 0$

Let $x = t$ **, therefore:** $y = \frac{4}{5}t$

Sub. $x = t$ **&** $y = \frac{4}{5}t$ **into (1) we get:** $z = \frac{11}{5}t$

$\vec{m} = \left[t, \frac{4}{5}t, \frac{11}{5}t\right] = \frac{1}{5}t[5,4,11]$

$\vec{m} = [5,4,11]$ **or** $[-5,-4,-11]$

Equation of a line with direction vector of $\vec{m}$ taht passes through point (5,0,0) is:

$\therefore \vec{r} = [5,0,0] + t[5,4,11] \quad , t \in R$

or $\vec{r} = [5,0,0] + t[-5,-4,-11]$

$\begin{cases} x = 5 + 5t \\ y = 4t \\ z = 11t \end{cases} \quad \textit{or} \quad \begin{cases} x = 5 - 5t \\ y = -4t \\ z = -11t \end{cases}$

9. Find, if possible, the value(s) of k such that the lines ℓ_1:[9,3,2]+t[3,k,−15] and ℓ_2:[−5,4,−2]+t[10,12,50] are:

a) parallel **No possible values for k**

Two lines are parallel iff their direction vectors are collinear:

[3,k,−15] = m [10,12,50]

3=10m (1)

k=12m (2)

-15=50m (3)

From (1) we get $m = \frac{3}{10}$ **from (3) we get** $m = -\frac{3}{10}$

This contradiction shows there is no such a k value that exists.

b) perpendicular **k=60**

$[3,k,-15]\bullet[10,12,50]=0$

$30+12k-750=0$

$12k=720$

$\boxed{k=60}$

10. Point P_1 lies on the line ℓ_1:[4,4,−3]+t[2,1,−1],t∈ℝ, and point P_2 lies on the line ℓ_2:[−2,−7,2]+s[3,2,−3]. If the vector $\overrightarrow{P_1P_2}$ is perpendicular to both ℓ_1 and ℓ_2, determine the coordinates of P_1 and P_2.

$P_1=(4+2t,4+t,-3-t)$ and $P_2=(-2+3s,-7+2s,2-3s)$

$\overrightarrow{P_1P_2}=[-6+3s-2t,-11+2s-t,5-3s+t]$

$\vec{m}_1\times\vec{m}_2=[2,1,-1]\times[3,2,-3]$

$=[-1,3,1]$

$\overrightarrow{P_1P_2}=k(\vec{m}_1\times\vec{m}_2)$

$[-6+3s-2t,-11+2s-t,5-3s+t]=k[-1,3,1]$

$-6+3s-2t=-k\longrightarrow 3s-2t+k=6\quad(1)$

$-11+2s-t=3k\longrightarrow 2s-t-3k=11\quad(2)$

$5-3s+t=k\longrightarrow 3s-t+k=5\quad(3)$

$(1)-(3):-t=-1$

$\boxed{t=1}$

$3s+k=4\quad(4)$

$2s-3k=10\quad(5)$

$3\times(4)+(5):\ 11s=22$

$\boxed{s=2}$ & $\boxed{k=-2}$

$P_1=(4+2t,4+t,-3-t)$ & $P_2=(-2+3s,-7+2s,2-3s)$

$P_1=(6,5,-4)$ $P_2=(4,-3,-4)$

3-7 Warm Up

1. Determine whether $l_1: \begin{matrix} x = 2 - t \\ y = 5t \end{matrix}$ and $l_2: x - 3 = \frac{1-y}{5}$ are coincident.

2. Develop vector, parametric, symmetric and Cartesian equations of the line through the point (3, -5) and is perpendicular to the line: $\begin{matrix} x = 3t - 5 \\ y = 2 + t \end{matrix}$.

3. Determine the Cartesian equation of the line passing through the point P(5, 4) and perpendicular to $\vec{u} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$.

Solutions:

① $l_1: (x,y) = (2,0) + t(-1,5)$

$\vec{d}_1 = [-1,5]$ $\quad$ $\vec{d}_2 = [1,-5] = -\vec{d}_1$ $\quad$ $\because \vec{d}_2 = -\vec{d}_1$ $\therefore$ lines are parallel

parametric equations of l_2: $t = x-3 \rightarrow x = 3+t$

$t = \frac{1-y}{5} \rightarrow y = 1-5t$

check if (2,0) lies on l_2: $\begin{matrix} 2=3+t \\ -1=t \end{matrix}$ $\quad$ $\begin{matrix} 0=1-5t \\ \frac{1}{5}=t \end{matrix}$ $\quad$ $\because$ the t-values are not the same

$\therefore$ (2,0) is not on l_2

$\therefore$ l_1 and l_2 are not coincident

② $\vec{d} = [3,1]$ $\quad$ $\vec{n} = [1,-3]$ or any scalar multiple

vector equation: $\vec{r} = (3,-5) + t(1,-3),\ t \in \mathbb{R}$

parametric equation: $\begin{matrix} x = 3+t \\ y = -5-3t \end{matrix}$, $t \in \mathbb{R}$

symmetric equation: $x-3 = \frac{y+5}{-3}$

Cartesian equation:

$x-3 = \frac{y+5}{-3}$

$-3x+9 = y+5$

$3x+y-4=0$

③ $\vec{n} = \begin{pmatrix} 2 \\ -6 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\therefore x - 3y + C = 0$

sub point (5,4) into equation:

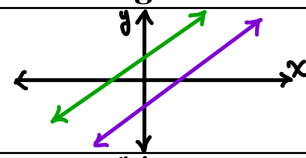
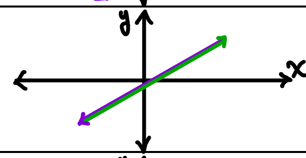
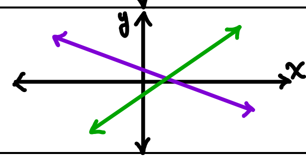
$(5) - 3(4) + C = 0$

$C = 7$

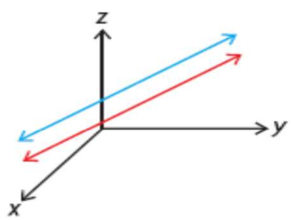
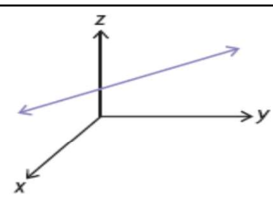
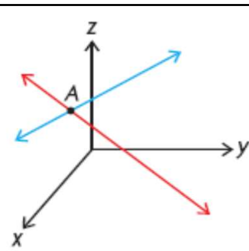
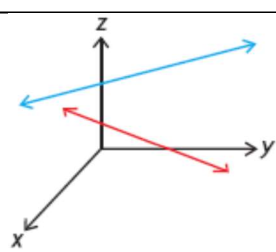
$\therefore$ Cartesian equation is $x-3y+7=0$

The Intersection of Two Lines in R^2 and R^3

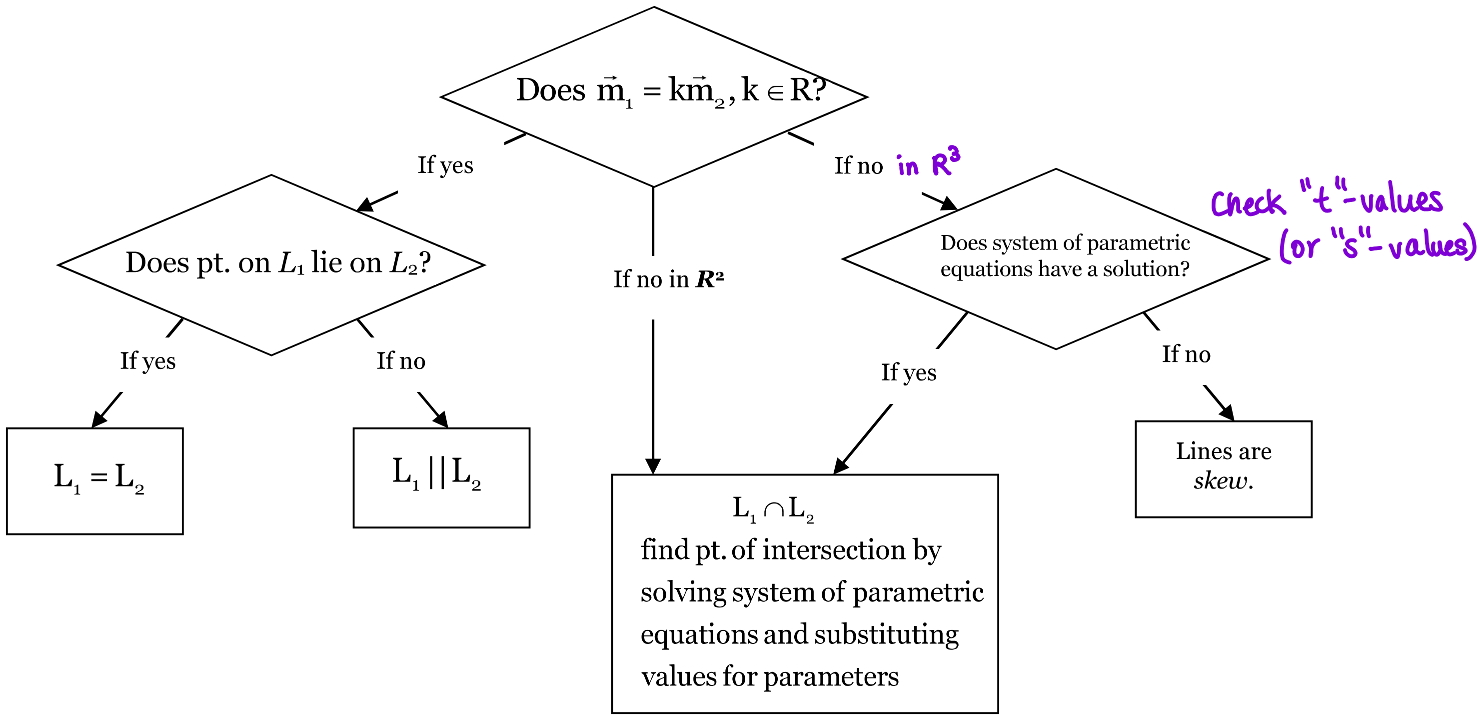
In R^2

Lines may…	Diagram	Number of solutions
be parallel (distinct) $\vec{m}_1 = k\vec{m}_2, k \in \mathbb{R}$		none
coincide (be coincident) $\vec{m}_1 = k\vec{m}_2, k \in \mathbb{R}$		infinite
intersect $\vec{m}_1 \neq k\vec{m}_2, k \in \mathbb{R}$		one

In $\boldsymbol{R^3}$ (the **Notation** is the same as for $\boldsymbol{R^2}$)

Lines may…	Conditions	Number of solutions
be parallel $\vec{m}_1 = k m_2, k \in \mathbb{R}$		none
coincide (be coincident) $\vec{m}_1 = k\vec{m}_2, k \in \mathbb{R}$		infinite
intersect (and are therefore coplanar) $\vec{m}_1 \neq k\vec{m}_2, k \in \mathbb{R}$		one
skew (do not intersect **and** are not parallel) $\vec{m}_1 \neq k\vec{m}_2, k \in \mathbb{R}$ but s and t-values are not the same for all components		none

Method for Determining Line Situation



Examples for R^2:

Are each of the following pairs of lines parallel, coincident or intersecting? If the lines intersect, find the point of intersection.

1) $L_1: 2x + 3y - 30 = 0$
$L_2: x - 2y + 13 = 0$

$\vec{m}_1 = [3, -2]$

$\vec{m}_2 = [-2, -1]$

$\vec{m}_1 \neq k\vec{m}_2$

$\therefore$ L_1 and L_2 not parallel

$\therefore$ L_1 and L_2 intersect

$L_1: 2x + 3y - 30 = 0$

$2 \times L_2: 2x - 4y + 26 = 0$

$L_1 - 2 \times L_2: \quad 7y - 56 = 0$

$7y = 56$

$y = 8$

sub $y = 8$ into L_2:

$x - 2(8) + 13 = 0$

$x = 3$

$\therefore$ Point of Intersection at $(3, 8)$.

2) $L_1 : (x, y) = (18, -2) + t(3, -2)$

$L_2 : (x, y) = (-5, 4) + s(2, 1)$

$\vec{m}_1 \neq k m_2$

$\therefore L_1$ and L_2 intersect

$L_1: x = 18 + 3t$

$y = -2 - 2t$

$L_2: x = -5 + 2s$

$y = 4 + s$

Equate x:

$18 + 3t = -5 + 2s$

$2s - 3t = 23 \rightarrow ①$

Equate y:

$-2 - 2t = 4 + s$

$s + 2t = -6 \rightarrow ②$

① - 2x②:

$-7t = 35$

$t = -5$

$(x, y) = (18, -2) + (-5)(3, -2)$

$= (3, 8)$

If solved for $s = 4$:

$(x, y) = (-5, 4) + (4)(2, 1)$

$= (3, 8)$

$\therefore$ Pt of Intersection is $(3, 8)$.

3) $L_1 : (x, y) = (1, 3) + t(4, 2)$

$L_2 : \dfrac{x - 2}{2} = y - 1$

$\vec{m}_1 = [4, 2]$

$\vec{m}_2 = [2, 1]$

$\because \vec{m}_1 = 2\vec{m}_2$

$\therefore L_1$ and L_2 are parallel

sub $(2, 1)$ from L_2 into L_1:

$[2, 1] = [1, 3] + t[4, 2]$

$2 = 1 + 4t$	$1 = 3 + 2t$
$1 = 4t$	$-2 = 2t$
$\frac{1}{4} = t$	$-1 = t$

$\because$ t-values are not the same, $\therefore$ the point is not on the line

$\therefore L_1$ and L_2 are parallel and distinct.

Examples for R^3:

Are each of the following pairs of lines parallel, coincident, intersecting or *skew* ? If the lines intersect, find the point of intersection.

4) $L_1 : (x, y, z) = (-1, 1, 0) + t(3, 4, -2)$
$L_2 : (x, y, z) = (-1, 0, -7) + s(2, 3, 1)$

$\vec{m}_1 = [3,4,-2]$
$\vec{m}_2 = [2,3,1]$

$\because \vec{m}_1 \neq k\vec{m}_2$
$\therefore$ lines will intersect

L_1: $x = -1+3t$
$y = 1+4t$
$z = -2t$

L_2: $x = -1+2s$
$y = 3s$
$z = -7+s$

Equate x: $-1+3t = -1+2s$
$2s-3t = 0 \rightarrow ①$

Equate y: $1+4t = 3s$
$3s-4t = 1 \rightarrow ②$

Equate z: $-2t = -7+s$
$s+2t = 7 \rightarrow ③$

$3\times③ - ②$: $10t = 20$
$t = 2$

sub $t=2$ into ③: $s+2(2) = 7$
$s = 3$

check $s=3$ and $t=2$ with ①:

$LS = 2s-3t$
$= 2(3)-3(2)$
$= 0$
$= RS$

$\therefore$ s and t satisfies all 3 components

Point of Intersection:

$(x,y,z) = (-1,1,0) + (2)(3,4,-2)$
$= (5,9,-4)$

5) $L_1 : (x,y,z) = (2,1,0) + t(1,-1,1)$

$L_2 : (x,y,z) = (3,0,-1) + s(2,3,-1)$

$\vec{m}_1 = [1,-1,1]$

$\vec{m}_2 = [2,3,-1]$

$\because \vec{m}_1 \neq k\vec{m}_2$

$\therefore$ lines are not parallel

$L_1: x = 2+t$
$y = 1-t$
$z = t$

$L_2: x = 3+2s$
$y = 3s$
$z = -1-s$

Equate x: $2+t = 3+2s$
$2s - t = -1 \rightarrow ①$

Equate y: $1-t = 3s$
$3s+t = 1 \rightarrow ②$

Equate z: $t = -1-s$
$s+t = -1 \rightarrow ③$

① + ② : $5s = 0$
$s = 0$

sub $s=0$ into ① :
$2(0) - t = -1$
$t = 1$

sub $s=0$, $t=1$ into ③:

$LS = s+t$
$= (0)+(1)$
$= 1$

$RS = -1$

$\because LS \neq RS$

$\because$ s and t do not satisfy all 3 equations

$\therefore$ There is no point of intersection and L_1 and L_2 are skew lines.

6) $L_1: x = 1 - 2s, y = s, z = -1 - s, s \in R$

$L_2: \dfrac{x+1}{-2} = \dfrac{1-y}{-1} = z - 2$

$\vec{m}_1 = [-2, 1, -1]$

$\vec{m}_2 = [-2, 1, 1]$

$\because \vec{m}_1 \neq k\vec{m}_2$

$\therefore$ lines are not parallel

$L_1: x = 1 - 2s$
$y = s$
$z = -1 - s$

$L_2: x = -1 - 2t$
$y = 1 + t$
$z = 2 + t$

Equate x: $1 - 2s = -1 - 2t$
$2s - 2t = 2 \rightarrow ①$

Equate y: $s = 1 + t$
$s - t = 1 \rightarrow ②$

Equate z: $-1 - s = 2 + t$
$s + t = -3 \rightarrow ③$

②+③: $2s = -2$
$s = -1$

sub $s = -1$ into ②: $(-1) - t = 1$
$t = -2$

check $s = -1, t = -2$ in ①:

$LS = 2s - 2t$
$= 2(-1) - 2(-2)$
$= 2$

$RS = 2$

$\therefore LS = RS$

Point of Intersection:

$x = 1 - 2(-1)$
$= 3$

$y = -1$

$z = -1 - (-1)$
$= 0$

$\therefore$ Point of intersection is $(3, -1, 0)$

Practice

1. Find the value(s) of a and b that make the lines $\ell_1 : \vec{r} = [3,0,-2] + t[3,1,-3]$, $\ell_2 : \vec{r} = [15,4,a] + s[5,b,-5]$

   a) Coincident
   b) Parallel and distinct
   c) Intersecting
   d) Skew

2. Determine the parametric equations of a line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x-4}{3} = \frac{y+1}{5} = \frac{z-4}{2}$ and $\frac{x}{6} = \frac{y-7}{10} = \frac{z+3}{5}$ and passes through the point (5,0,−2).

3. Find the vector equation of the line through the point (8,10,10) that meets the line $\frac{x+8}{-1} = \frac{y-11}{3} = \frac{z-1}{4}$ at 90° angles.

4. Lines $\ell_1 :\ \vec{r} = [2,1,3] + t[6,-4,-1]; t \in R$ and $\ell_2 : \begin{cases} x-1 = -s \\ y-6 = \mathbf{a}s \\ z-2 = \mathbf{b}s \end{cases}$ are intersecting at point (-1,3,2). What are the possible values of **a** and **b**?

5. Find all values of ***k*** for which the following lines **do not** intersect.

$$\ell_1 : \begin{cases} x = -1 + 2r \\ y = 3k + r \\ z = 1 + 3r \end{cases} \quad \text{and} \quad \ell_2 : \vec{r} = [1,0,-2] + t[-2,3,1]$$

6. Determine the point(s) of intersection between line $x-2 = t, y+2 = 3t, 2+z = 2t, t \in \mathbb{R}$ and sphere with equation $x^2 + y^2 + z^2 = 100$.

7. Determine if the following lines are parallel, skew or intersecting. In the case the lines are intersecting, find the point of intersection.
   L_1: [x, y, z] = [-3, 1, 4] + t[1,-1,-4] and L_2: [x, y, z) = [1,4,6] + s[6,1,7]

8. Determine why the lines $\vec{r} = [1,3,4] + s[2,3,5]$ and $\vec{v}$ = [1,1,1]+ t[2, 2,-2]are not perpendicular.

MCV4UE The Intersection of Two Lines in R^2 and R^3

Practice-Solution

1. Find the value(s) of a and b that make the lines $\begin{array}{l} \ell_1 : \vec{r} = [3,0,-2] + t[3,1,-3] \\ \ell_2 : \vec{r} = [15,4,a] + s[5,b,-5] \end{array}$

   a) Coincident
   b) Parallel and distinct
   c) Intersecting
   d) Skew

a) $L_1 = L_2 \Leftrightarrow \overrightarrow{m_1} = k\overrightarrow{m_2}$ & a point belongs to L_1 also belongs to L_2

$\overrightarrow{m_1} = [3,1,-3]$

$\overrightarrow{m_2} = [5,b,-5]$

$\overrightarrow{m_1} = k\overrightarrow{m_2} \Rightarrow [3,1,-3] = k[5,b,-5]$

$$3 = 5k \quad \uparrow k = \frac{3}{5}$$

$$1 = bk \xrightarrow{k=\frac{3}{5}} \boxed{b = \frac{5}{3}}$$

$$-3 = -5k$$

$$[3,0,-2] = [15,4,a] + s\left[5, \frac{5}{3}, -5\right]$$

$$3 = 15 + 5s \quad \uparrow s = \frac{-12}{5}$$

$$0 = 4 + \frac{5}{3}s$$

$$-2 = a - 5s \xrightarrow{s=\frac{-12}{5}} a = -2 + 5\left(\frac{-12}{5}\right)$$

$$\boxed{a = -14}$$

b) $a \neq -14, b = \frac{5}{3}$

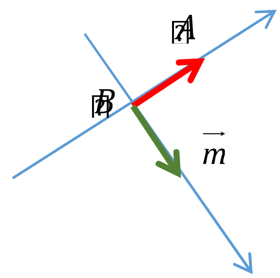
c) $a = -14, b \neq \frac{5}{3}$

d) $a \neq -14, b \neq \frac{5}{3}$

2. Determine the parametric equations of a line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x-4}{3}=\frac{y+1}{5}=\frac{z-4}{2}$ and $\frac{x}{6}=\frac{y-7}{10}=\frac{z+3}{5}$ and passes through the point (5,0,−2).

$$\overrightarrow{m_1}=[3,5,2] \text{ and } \overrightarrow{m_2}=[6,10,5]$$
$$\vec{m}=\overrightarrow{m_1}\times\overrightarrow{m_2}=[5,-3,0] \quad \text{point}(5,0,-2)$$
$$\begin{cases} x=5+5t \\ y=-3t \\ z=-2 \end{cases}$$

3. Find the vector equation of the line through the point (8,10,10) that meets the line $\frac{x+8}{-1}=\frac{y-11}{3}=\frac{z-1}{4}$ at 90° angles.



Let B(-8-t,11+3t,1+4t) be a point lies on the line L, and A(8,10,10). If $\vec{d}$ represents the direction vector of line perpendicular to L, we have:

$$\vec{d}=\overrightarrow{AB}=[-16-t,1+3t,4t-9].$$
$$\text{Since } \vec{d}\bullet\vec{m}=0 \Rightarrow [16+t,1-3t,9-4t]\bullet[-1,3,4]=0$$
$$-16-t+3+9t-36+16t=0$$
$$26t-17=0$$
$$t=\frac{17}{26}$$
$$\vec{d}=\left[\frac{-433}{26},\frac{77}{26},\frac{-166}{26}\right]=\frac{1}{26}[-433,77,-166]$$
$$\vec{r}=[8,10,10]+t[-433,77,-166]$$

4. Lines $l_1:\ \vec{r}=[2,1,3]+t[6,-4,1];t\in\mathbb{R}$ and $l_2:\ \begin{cases} x-1=-s \\ y-6=as \\ z-2=bs \end{cases}$ are intersecting at point (-1,3,2). What are the possible values of a and b ?

$$x=2+6t$$
$$-1=2+6t$$
$$t=\frac{-1}{2}$$

$$x=1-s \xrightarrow{\text{sub. } x=-1} -1=1-s \quad \text{or} \quad \boxed{s=2}$$
$$y=6+as \rightarrow 3=6+a(2) \quad \text{or} \quad \boxed{a=\frac{-3}{2}}$$
$$z=2+bs \rightarrow 2=2+b(2) \quad \text{or} \quad \boxed{b=0}$$

5. Find all values of ***k*** for which the following lines **do not** intersect.

$$l_1: \begin{cases} x = -1 + 2r \\ y = 3k + r \\ z = 1 + 3r \end{cases} \quad \text{and} \quad l_2: \vec{r} = [1,0,-2] + t[-2,3,1]$$

$$-1 + 2r = 1 - 2t \longrightarrow r + t = 1 \quad (1)$$
$$3k + r = 3t \longrightarrow -r + 3t = 3k \quad (2)$$
$$1 + 3r = -2 + t \longrightarrow 3r - t = -3 \quad (3)$$
$$(1) + (3): 4r = -2$$
$$r = \frac{-1}{2} \ \& \ t = \frac{3}{2}$$
$$\text{sub. } r = \frac{-1}{2} \ \& \ t = \frac{3}{2} \text{ into (3)}: \frac{1}{2} + \frac{9}{2} = 3k$$
$$5 = 3k$$
$$k = \frac{5}{3}$$

$\therefore$ **The given lines do not intersect iff** $k \neq \frac{5}{3}$

6. Determine the point(s) of intersection between line $x - 2 = t, y + 2 = 3t, 2 + z = 2t, t \in \mathbb{R}$ and sphere with equation $x^2 + y^2 + z^2 = 100$.

$$x = 2 + t$$
$$y = -2 + 3t$$
$$z = -2 + 2t$$
$$x^2 + y^2 + z^2 = 100$$
$$(2 + t)^2 + (-2 + 3t)^2 + (-2 + 2t)^2 = 100$$
$$4 + 4t + t^2 + 4 - 12t + 9t^2 + 4 - 8t + 4t^2 = 100$$
$$14t^2 - 16t - 88 = 0$$
$$7t^2 - 8t - 44 = 0$$
$$(7t - 22)(t + 2) = 0$$
$$\boxed{t = \frac{22}{7}} \text{ or } \boxed{t = -2}$$

$t = \frac{22}{7}$

$x = 2 + \frac{22}{7} \longrightarrow x = \frac{36}{7}$

$y = -2 + 3(\frac{22}{7}) \longrightarrow y = \frac{52}{7}$

$z = -2 + 2(\frac{22}{7}) \longrightarrow z = \frac{30}{7}$

$\therefore (\frac{36}{7}, \frac{52}{7}, \frac{30}{7})$

$t = -2$

$x = 2 + (-2) \longrightarrow x = 0$

$y = -2 + 3(-2) \longrightarrow y = -8$

$z = -2 + 2(-2) \longrightarrow z = -4$

$\therefore (0, -8, -4)$

7. Determine if the following lines are parallel, skew or intersecting. In the case the lines are intersecting, find the point of intersection.

L_1: [x, y, z] = [-3, 1, 4] + t[1,-1,-4] and L_2: [x, y, z) = [1,4,6] + s[6,1,7]

$x = -3 + t \qquad x = 1 + 6s$

$y = 1 - t \qquad y = 4 + s$

$z = 4 - 4t \qquad z = 6 + 7s$

$-3 + t = 1 + 6s \longrightarrow t - 6s = 4 \quad (1)$

$1 - t = 4 + s \longrightarrow -t - s = 3 \quad (2)$

$4 - 4t = 6 + 7s \longrightarrow 4t + 7s = -2 \quad (3)$

$(1) + (2): -7s = 7$

$s = -1 \ \& \ t = -2$

Check: $4t + 7s = -2$

$4(-2) + 7(-1) \neq -2$

$\therefore$ Two lines are skew

8. Determine why the lines $\vec{r} = [1,3,4] + s[2,3,5]$ and $\vec{v} = [1,1,1] + t[2, 2,-2]$ are not perpendicular.

Although [2,3,5]•[2,2,-2]=0, two lines are not perpendicular because they are skew!(Prove it!)

3-8 Warm Up

1. Prove that the lines $l_1: (x-1, y-4, z) = s(-4, 2, 6), s \in \mathbb{R}$ and $l_2: (x, y, z) = (-3, 3, 0) + t(0, 1, 2),\ t \in \mathbb{R}$ lie in the same plane.

Solutions:

① $l_1: (x,y,z) = (1,4,0) + s(-4,2,6)$
$\vec{d}_1 = (-4,2,6)$

$l_2: (x,y,z) = (-3,3,0) + t(0,1,2)$
$\vec{d}_2 = (0,1,2)$

$\because \vec{d}_1 \neq k\vec{d}_2 \therefore$ non-parallel lines

l_1: $x = 1-4s$, $y = 4+2s$, $z = 6s$

l_2: $x = -3$, $y = 3+t$, $z = 2t$

$\longrightarrow$
$1-4s = -3$ ①
$4+2s = 3+t$ ②
$6s = 2t$ ③

From ①: $s = 1$

sub into ②: $4+2(1) = 3+t$
$3 = t$

sub $s=1, t=3$ into ③:

$LS = 6(1) = 6$ $\quad$ $RS = 2(3) = 6$

$\because s=1$ and $t=3$ satisfies all 3 equations, there is one point of intersection and the lines lie on the same plane

Extension: Find the point of intersection:

sub $t=3$ into l_2:

$x = -3$
$y = 3+(3) = 6$
$z = 2(3) = 6$

$\therefore$ Point of Intersection is $(-3,6,6)$

1. Write each of the following lines in scalar, vector, parametric, and symmetric form.

Scalar (Cartesian)	Vector	Parametric	Symmetric
$2x + 3y - 6 = 0$	$pt = (0,2)$, $\vec{m} = [3,-2]$ $\vec{r} = [0,2] + t[3,-2]$	$\begin{cases} x = 3t \\ y = 2 - 2t \end{cases}$	$\frac{x}{3} = \frac{y-2}{-2}$
	$pt = (2,2)$, $\vec{m} = [2, \frac{3}{2}] = \frac{1}{2}[4,3]$ $3x - 4y + 2 = 0$	$\begin{cases} x = 2t + 2 \\ y = \frac{3}{2}t + 2 \end{cases}$	$\frac{x-2}{2} = \frac{4-2y}{-3}$
	$\vec{r} = [2,-3,1] + t[1,-1,4]$	$\begin{cases} x = 2 + t \\ y = -3 - t \\ z = 1 + 4t \end{cases}$	$x - 2 = \frac{y+3}{-1} = \frac{z-1}{4}$
	$\vec{r} = [0,-1,2] + t[-2,3,3]$	$\begin{cases} x = -2t \\ y = -1 + 3t \\ z = 3t + 2 \end{cases}$	$\frac{x}{-2} = \frac{y+1}{3} = \frac{z-2}{3}$

2. Determine the **exact** value(s) of k that would make the following lines intersect at 60° angle.

$$L_1: \vec{r} = [17,-16] + t[1,k] \text{ and } L_2: \frac{x-25}{1} = \frac{y-3}{-1}$$

$$\cos(60^\circ) = \frac{\vec{m}_1 \bullet \vec{m}_2}{|\vec{m}_1||\vec{m}_2|}$$

$$\frac{1}{2} = \frac{[1,k]\bullet[1,-1]}{\left(\sqrt{1+k^2}\right)\left(\sqrt{2}\right)}$$

$$\frac{1}{2} = \frac{1-k}{\left(\sqrt{1+k^2}\right)\left(\sqrt{2}\right)}$$

$$\sqrt{2(1+k^2)} = 2(1-k) \quad (k < 1)$$

$$2(1+k^2) = 4(1-k)^2$$

$$1 + k^2 = 2(1 - 2k + k^2)$$

$$1 + k^2 = 2 - 4k + 2k^2$$

$$k^2 - 4k + 1 = 0$$

$$k = 2 \pm \sqrt{3}$$

$$\boxed{k = 2 - \sqrt{3}}$$

3. Determine parametric equations of a line that is parallel to $\vec{r}_1 = \left[3, \frac{5}{2}, -5\right] + t[-1,1,0]$ and passes through the x-intercept of $\begin{cases} x = 2 + 3t \\ y = \frac{1}{2} + \frac{1}{4}t \\ z = 6 + 3t \end{cases}$.

$\vec{m} == [-1,1,0]$

x - int : $y = z = 0$

$0 = \frac{1}{2} + \frac{1}{4}t \rightarrow t = -2$

$0 = 6 + 3t \rightarrow t = -2$

$x = 2 + 3(-2)$

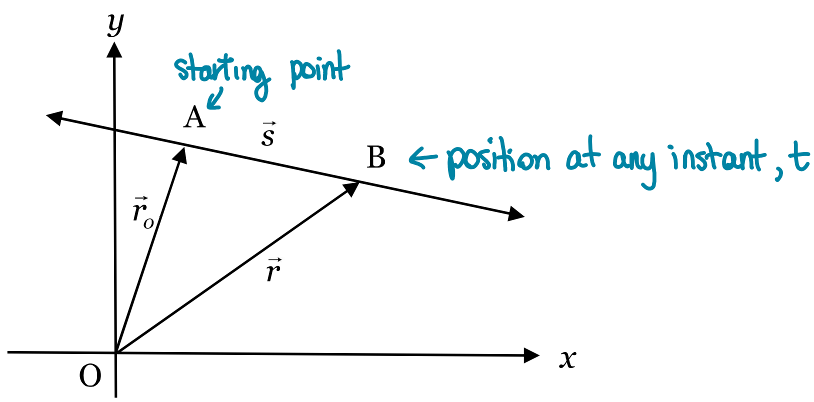
$\quad = -4$

x - int : $(-4,0,0)$

$$\begin{cases} x = -4 - t \\ y = t \\ z = 0 \text{ , } t \in \mathbb{R} \end{cases}$$

Application to Constant Motion in Two-Dimensional Space

Consider an object moving with constant velocity , $\vec{v}$, modeled as a point particle moving along an arbitrary path in the *xy*-plane. We assume that we are able to detect the particle's position at any point and to measure the corresponding clock time. Two positions *A* and *B* in the particle's path are shown. Let the vectors that locate these positions be $\vec{r}_0$ and $\vec{r}$, respectively. The displacement , $\vec{s}$, can be expresses as $\vec{s} = \vec{r} - \vec{r}_o$.   Recall: displacement = velocity × time



Therefore, $displacement = \vec{s} = \overrightarrow{AB}$

$= \vec{r} - \vec{r}_o$

Since $\vec{s} = t\vec{v}$, *where* $\vec{v}$ *is the velocity*

$\vec{r} - \vec{r}_o = t\vec{v}$

or $\boxed{\vec{r} = \vec{r}_o + t\vec{v}}$ ← vector equation $\vec{r} = \vec{r}_0 + t\vec{m}$

Suppose that $\vec{r}_o = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ km and the cyclist's velocity is $\begin{pmatrix} 10 \\ -2 \end{pmatrix}$ km/h. B is an arbitrary position on the cyclist's path, so $\overrightarrow{OB} = \vec{r}$ can be written as $\begin{pmatrix} x \\ y \end{pmatrix}$ so that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 10 \\ -2 \end{pmatrix}$ for any value of t ≥ 0. The position vector of the cyclist can be found at any time by replacing *t* with the appropriate numerical value.

Example 1: An object, P, moves in a straight line with constant velocity. Its position vector, relative to an origin, O, is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}, t \geq 0$, where *t* is measured in hours and displacement is measured in kilometers

(a) Find the coordinates of P   (i) initially   (ii) after 5 hours.

i) $t = 0$

$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} + 0\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 9 \end{pmatrix}$

∴ Coordinates of P are (5,9)

ii) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} + 5\begin{pmatrix} 4 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 9 \end{pmatrix} + \begin{pmatrix} 20 \\ -15 \end{pmatrix}$

$= \begin{pmatrix} 25 \\ -6 \end{pmatrix}$

∴ Coordinates of P after 5h are (25,-6)

Another object, Q, also moves in a straight line with constant velocity so that its position vector at time, t, is given $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \end{pmatrix}, t \geq o$

(b) Show that the paths of P and Q intersect, but P and Q do not collide.

$\vec{d}_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad \vec{d}_2 = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

$\because \vec{d}_1 \neq k\vec{d}_2$

$\therefore$ paths of P and Q are not parallel and they intersect at N. Suppose that P reaches N at time t_1 and Q reaches N at time t_2.

P: $x = 5 + 4t_1$ $\quad$ Q: $x = -3 + 5t_2$

$y = 9 - 3t_1$ $\quad$ $y = 4 - t_2$

$\therefore$ $5 + 4t_1 = -3 + 5t_2 \Rightarrow 4t_1 - 5t_2 = -8$ —— ①

$9 - 3t_1 = 4 - t_2 \qquad 3t_1 - t_2 = 5$ —— ②

Solving ① and ②:

① − 5×② : $4t_1 - 5t_2 = -8$

$15t_1 - 5t_2 = 25$

$-11t_1 = -33$

$t_1 = 3$

Sub in ① : $4(3) - 5t_2 = -8$

$12 - 5t_2 = -8$

$-5t_2 = -20$

$t_2 = 4$

$\therefore$ P reaches N after 3 hours and Q reaches N after 4 hours. Hence they do not collide.

Example 2. A cyclist is traveling at a speed of 26 km/h in a direction $\begin{pmatrix} 12 \\ -5 \end{pmatrix}$ relative to an origin, O

She starts at point A(-2, 10) and, after one hour she has reached point B

(a) Write down a unit vector parallel to the cyclist's velocity and use it to find her velocity as a column vector

$$\left|\begin{pmatrix} 12 \\ -5 \end{pmatrix}\right| = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$$

$\therefore$ A unit vector parallel to the cyclist's velocity is $\begin{pmatrix} \frac{12}{13} \\ \frac{-5}{13} \end{pmatrix}$.

Recall : $\hat{v} = \dfrac{\vec{v}}{|\vec{v}|}$

If she travels at a speed of 26km/h, then her velocity is :

$$26\begin{pmatrix} \frac{12}{13} \\ \frac{-5}{13} \end{pmatrix} = \begin{pmatrix} 24 \\ -10 \end{pmatrix} \text{ km/h}$$

check : $\left|\begin{pmatrix} 24 \\ -10 \end{pmatrix}\right| = \sqrt{24^2 + (-10)^2} = 26$

(b) Find (i) $\overrightarrow{OA}$ (ii) $\overrightarrow{AB}$ (iii) $\overrightarrow{OB}$

i) $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$

ii) $\overrightarrow{AB} = \begin{pmatrix} 24 \\ -10 \end{pmatrix}$

iii) $\vec{r} = \begin{pmatrix} -2 \\ 10 \end{pmatrix} + t\begin{pmatrix} 24 \\ -10 \end{pmatrix}$

Sub t = 1,

$\vec{r} = \begin{pmatrix} 22 \\ 0 \end{pmatrix}$

or $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$\overrightarrow{OB} = \begin{pmatrix} -2 \\ 10 \end{pmatrix} + \begin{pmatrix} 24 \\ -10 \end{pmatrix}$

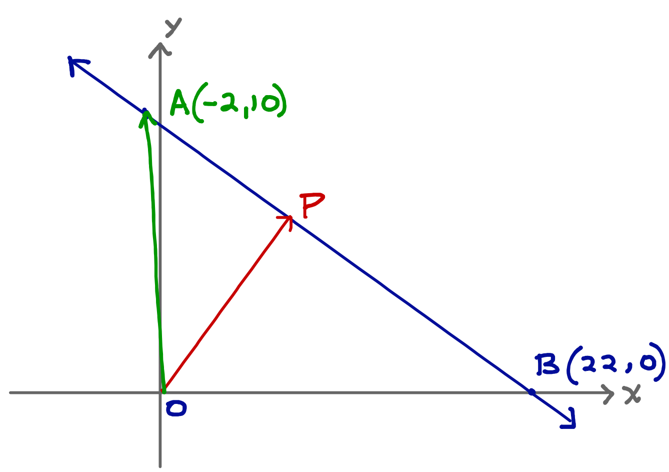
$\overrightarrow{OB} = \begin{pmatrix} 22 \\ 0 \end{pmatrix}$

(c) Find the coordinates of B

The coordinates of B are (22, 0)

After t hours, she is at point P.

(d) On Cartesian axes, show the cyclist's path and the points O, A B and an arbitrary point P.



(e) Find (i) $\overrightarrow{AP}$ (ii) $\overrightarrow{OP}$ (iii) $\overrightarrow{OP} \bullet \overrightarrow{AP}$ in terms of t

i) $\overrightarrow{AP} = t\overrightarrow{AB}, t \in \mathbb{R}, t \geq 0$

$= t\begin{pmatrix} 24 \\ -10 \end{pmatrix}$

$= \begin{pmatrix} 24t \\ -10t \end{pmatrix}$

ii) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$= \begin{pmatrix} -2 \\ 10 \end{pmatrix} + \begin{pmatrix} 24t \\ -10t \end{pmatrix}$

$= \begin{pmatrix} -2+24t \\ 10-10t \end{pmatrix}$

iii) $\overrightarrow{OP} \cdot \overrightarrow{AP} = \begin{pmatrix} -2+24t \\ 10-10t \end{pmatrix} \cdot \begin{pmatrix} 24t \\ -10t \end{pmatrix}$

$= 24t(-2+24t) + (-10t)(10-10t)$

$= -48t + 576t^2 - 100t + 100t^2$

$= 676t^2 - 148t$

(f) Hence, find the time, to the nearest minute, for the cyclist to be closest to O and the distance $|\overrightarrow{OP}|$ at this time.

The cyclist is closest to O when $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{AP}$, i.e. $\overrightarrow{OP} \cdot \overrightarrow{AP} = 0$

$\therefore 676t^2 - 148t = 0$

$t(676t - 148) = 0$

$t = 0$ or $676t - 148 = 0$

inadmissible since it's the starting point

$t = \frac{148}{676}$

$t = 0.21893\ h$

$t \approx 13$ min

Sub $t = 0.21893$ into $\overrightarrow{OP}$

$\overrightarrow{OP} = \begin{pmatrix} -2+24t \\ 10-10t \end{pmatrix}$

$= \begin{pmatrix} -2+24(0.21893) \\ 10-10(0.21893) \end{pmatrix}$

$= \begin{pmatrix} 3.25432 \\ 7.8107 \end{pmatrix}$

$|\overrightarrow{OP}| = \sqrt{(3.25432)^2 + (7.8107)^2}$

≈ 8.46

$\therefore$ The distance OP at this time is ~ 8.46 km.

Practice

1. The position of two submarines S_1 and S_2 at time t hours are given by the formulas
   S_1: $[x, y, z] = [2, 1, -4] + t\,[2, 1, 2]$
   S_2 : $[x, y, z] = [1, 1, 1] + t\,[2, 0.5, -4]$

   (a) What is the speed of the first submarine?
   (b) Determine if the paths of the two submarines will intersect.
   (c) Determine if the two submarines will collide.

2. Position in km of a helicopter is given by $\mathbf{r} = \begin{pmatrix} 17 \\ -11 \end{pmatrix} + t\begin{pmatrix} -20 \\ 21 \end{pmatrix}$ where 't' is the number of hours after 8:00 a.m. Sherwood Park is at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find:

   (a) distance from Sherwood Park at 10:00 a.m.
   (b) time when the plane is 123 km west and 133 km north of Sherwood Park

3. A particle is moving with a constant velocity along line L. Its initial position is A(6, –2, 10) and after one second it has moved to B(9, –6, 15).
   (a) Find the velocity vector $\overrightarrow{AB}$ and find the speed of the particle.
   (b) Write down a possible vector equation of the line L.

4. The position of ship A is given by $\mathbf{a} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and ship B by $\mathbf{b} = \begin{pmatrix} -13 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ where the distance is in kilometres and t is the number of hours after 9:00 A.M. The base is at the origin. Find:
   (a) the position of ship A at 1:00 p.m. relative to the base.
   (b) distance between the 2 ships at 1:00 p.m.
   (c) the time when they would collide.
   (d) the speed of ship A.

Practice

1. The position of two submarines S_1 and S_2 at time t hours are given by the formulas
   S_1: $[x, y, z] = [2, 1, -4] + t\,[2, 1, 2]$
   S_2 : $[x, y, z] = [1, 1, 1] + t\,[2, 0.5, -4]$

   (a) What is the speed of the first submarine?

   $$\text{speed} = \sqrt{2^2 + 1^2 + 2^2} = 3$$

   (b) Determine if the paths of the two submarines will intersect.

   $$x = 2 + 2t \qquad x = 1 + 2s$$
   $$y = 1 + t \qquad y = 1 + 0.5s$$
   $$z = -4 + 2t \qquad z = 1 - 4s$$
   $$2 + 2t = 1 + 2s \longrightarrow 2t - 2s = -1 \quad (1)$$
   $$1 + t = 1 + 0.5s \longrightarrow t - 0.5s = 0 \quad (2)$$
   $$-4 + 2t = 1 - 4s \longrightarrow 2t + 4s = 5 \quad (3)$$
   $$(2) - (1): 6s = 6$$
   $$s = 1 \ \& \ t = 0.5$$

   The path of two submarines will intersect

   (c) Determine if the two submarines will collide.

   The two submarines will not collide since the s and t are different.

2. Position in km of a helicopter is given by $\mathbf{r} = \begin{pmatrix} 17 \\ -11 \end{pmatrix} + t\begin{pmatrix} -20 \\ 21 \end{pmatrix}$ where t is the number of hours after 8:00 a.m. Sherwood Park is at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find:

   (a) distance from Sherwood Park at 10:00 a.m.

   $$\mathbf{r} = \begin{pmatrix} 17 \\ -11 \end{pmatrix} + 2\begin{pmatrix} -20 \\ 21 \end{pmatrix} = \begin{pmatrix} -23 \\ 31 \end{pmatrix}$$

   $$|\mathbf{r}| = \sqrt{(-23)^2 + 31^2} = 38.6$$

   (b) time when the plane is 123 km west and 133 km north of Sherwood Park.

   $$\begin{pmatrix} 17 \\ -11 \end{pmatrix} + t\begin{pmatrix} -20 \\ 21 \end{pmatrix} = \begin{pmatrix} -123 \\ 133 \end{pmatrix}$$
   $$17 - 20t = -123 \longrightarrow t = 7$$
   $$-11 + 21t = 133 \longrightarrow t = 6.87$$

   This means the plane is NEVER 123 west and 133 north of Sherwood Park

3. A particle is moving with a constant velocity along line L. Its initial position is A(6, −2, 10) and after one second it has moved to B(9, −6, 15).

(a) Find the velocity vector $\overrightarrow{AB}$ and find the speed of the particle.

$$\overrightarrow{\mathbf{AB}} = \overrightarrow{\mathbf{OB}} - \overrightarrow{\mathbf{OA}}$$

$$= \begin{pmatrix} 9-6 \\ -6+2 \\ 15-10 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$$\textbf{Speed} = \sqrt{3^2 + (-4)^2 + 5^2}$$

$$= 5\sqrt{2}$$

(b) Write down a possible vector equation of the line L.

$$\vec{b} = \begin{pmatrix} -13 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} -9 \\ 18 \end{pmatrix}$$

4. The position of ship A is given by $\mathbf{a} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ and ship B by $\mathbf{b} = \begin{pmatrix} -13 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ where the distance is in kilometres and t is the number of hours after 9:00 A.M. The base is at the origin. Find:

(a) the position of ship A at 1:00 p.m. relative to the base.

$$\vec{\mathbf{a}} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} + 4\begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 19 \end{pmatrix}$$

6 km west and 19 km north of base

(b) distance between the 2 ships at 1:00 p.m.

$$\vec{r} = \begin{pmatrix} 6 \\ -2 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} \text{ or } \vec{r} = \begin{pmatrix} 9 \\ -6 \\ 15 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$$\left|\vec{\mathbf{b}} - \vec{\mathbf{a}}\right| = \left|\begin{pmatrix} -9 \\ 18 \end{pmatrix} - \begin{pmatrix} -6 \\ 19 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} -3 \\ -1 \end{pmatrix}\right|$$

$$= \sqrt{(-3)^2 + (-1)^2} = \sqrt{10} \text{ km}$$

(c) the time when they would collide.

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -13 \\ 2 \end{pmatrix} + s\begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$t = s = 5$$

At 2:00 p.m

(d) the speed of ship A.

$$\text{Speed} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}\ \text{km/h}$$

5. Toy airplanes fly in a straight line at a constant speed. Airplane 1 passes through a point A.

Its position, t seconds after it has passed through A, is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$.

Find the speed of the airplane in ms^{-1}.

choosing velocity vector $\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$

$$\text{Speed} = \sqrt{4+9+1} = \sqrt{14} \doteq 3.74\ \text{m/s}$$

After seven seconds the airplane passes through a point B. Find the distance the airplane has travelled during the seven seconds.

$$B = (-11, 17, 7)$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -14 \\ 21 \\ 7 \end{pmatrix}$$

$$\text{distance} = \sqrt{(-14)^2 + 21^2 + 7^2}$$

$$= 7\sqrt{14} \doteq 26.2\text{m}$$

Airplane 2 passes through a point C. Its position q seconds after it passes through C is given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 8 \end{pmatrix} + q\begin{pmatrix} -1 \\ 2 \\ a \end{pmatrix}; q \in \mathbb{R}$. The angle between the flight paths of Airplane 1 and Airplane 2 is 60^{o}. Find the value(s) of a.

$$\cos(60^{o}) = \frac{[-2,3,1]\bullet[-1,2,a]}{\left(\sqrt{14}\right)\left(\sqrt{a^2+5}\right)}$$

$$\frac{1}{2} = \frac{8+a}{\sqrt{14(a^2+5)}}$$

$$\sqrt{14(a^2+5)} = 2(a+8) \quad (a > -8)$$

$$14(a^2+5) = 4(a^2+16a+64)$$

$$7a^2+35 = 2a^2+32a+128$$

$$5a^2-32a-93 = 0$$

$$\boxed{a = -2.17} \text{ or } \boxed{a = 8.57}$$

6. Ryan and Jack have model airplanes, which take off from level ground. Jack's airplane takes off after Ryan's. The position of Ryan's airplane seconds after it takes off is given by

$$\vec{r} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

a) Find the speed of Ryan's airplane

$$\text{Speed} = \sqrt{16+4+16} = \sqrt{36} = 6 \text{ m/s}$$

b) Find the height of Ryan's airplane after two seconds.

$$\vec{r} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 8 \end{pmatrix}$$

$$\text{height} = 8 \text{ m}$$

The position of Jack's airplane s seconds after it takes off is given by $\vec{r} = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$.

Show that the paths of the airplanes are perpendicular.

$$[-4,2,4]\bullet[4,-6,7]$$
$$= (-4\times4)+(2\times-6)+(4\times7)$$
$$= 0$$

The two airplanes collide at the point (-23,20,28).

How long after Ryan's airplane takes off does Jack's airplane take off?

$$\begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$$

$5 - 4t = -39 + 4s,\ 6 + 2t = 44 - 6s,\ 4t = 7s$

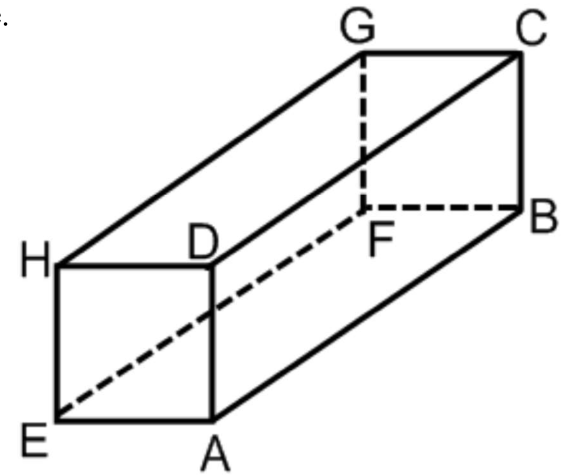
$t = 7$

$s = 4$

$\therefore$ **3 seconds later**

Mid-Review: Geometric Vector

1. The diagram shows a parallelepiped. Determine a single vector (with head and tail on the parallelepiped) that is equivalent to each sum or difference.

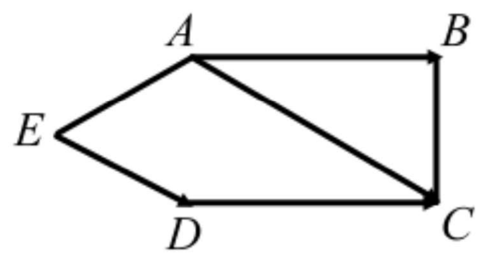


   $a)\quad -\overrightarrow{AE}+\overrightarrow{DA}=$ ________

   $b)\quad \overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CG}=$ _________

   $c)\quad \overrightarrow{FG}-\overrightarrow{DC}-\overrightarrow{AE}+\overrightarrow{AF}=$ ______

2. ABCDE is a pentagon such that $\overrightarrow{AB}=\overrightarrow{DC}$ and $\overrightarrow{AC}=2\overrightarrow{ED}$ .write each vector in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



   a) $\overrightarrow{EC}=$

   b) $\overrightarrow{CB}=$

   c) $\overrightarrow{AE}=$

3. If $\vec{a}$ and $\vec{b}$ are unit vectors that make an angle of 60^0 with each other, calculate $\left|3\vec{a}-4\vec{b}\right|$.

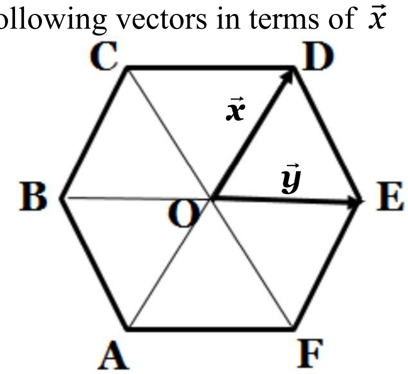
4. If $\frac{2}{3}\vec{x}=\vec{a}+\frac{1}{3}\vec{b}$, $2\vec{y}=-3\vec{a}+\vec{b}$ express $2\vec{a}-5\vec{b}$ in terms of $\vec{x},\vec{y}$.

5. Given that

$$\vec{u}=x\vec{a}+2y\vec{b}$$
$$\vec{v}=-2y\vec{a}+3y\vec{b}$$
$$\vec{w}=4\vec{a}-2\vec{b}$$

   where $\vec{a}$ and $\vec{b}$ are not collinear, find the values of x and y for which $2\vec{u}-\vec{v}=\vec{w}$.

6. Using the regular hexagon ABCDEF shown, express each of the following vectors in terms of $\vec{x}$ and $\vec{y}$.



   a) $\overrightarrow{\boldsymbol{DA}}=$
   b) $\overrightarrow{\boldsymbol{DE}}=$
   c) $\overrightarrow{\boldsymbol{BF}}=$

Mid-Review: Geometric Vector

7. Given $\vec{p} = [2, -3], \vec{q} = [-1, 4]$, evaluate $\left|3\vec{p} - 2\vec{q}\right|$.

8. Given the point $P(4, -3)$ where $\overrightarrow{PQ} = [7, -4]$ find

   a) coordinates of Q b) $\left|\overrightarrow{PQ}\right|$ c) a unit vector in the direction of $\overrightarrow{QP}$

9. If $\vec{u}$ =[1,4,-2], $\vec{v} = -2\hat{i} - 3\hat{j}$ and $\vec{w}$ =[-1,-3,1], find:

   a) $\left|3\vec{v} + 3\hat{i} - 2k\right|$

   b) a unit vector with the same direction as $\vec{u}$.

   c) Find the angle between $\vec{v}$ and $\vec{w}$

10. The points A(-1,2,-1) ,B(2,-1,3), and D(-3,1,-3) are three vertices of parallelogram ABCD. Find the coordinate of C.

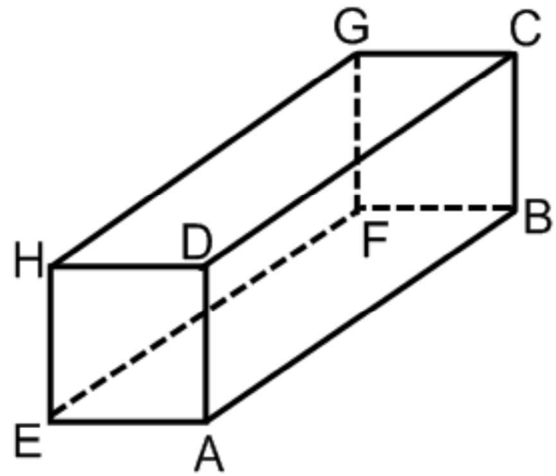
11. Vectors [2, -a, 1] and [−2, 2, -a+1] are collinear. Find the value of a.

12. The vectors $\vec{u}$ and $\vec{v}$ have lengths 2 and 1 respectively. The vectors $\vec{u} + 5\vec{v}$ and $2\vec{u} - 3\vec{v}$ are perpendicular. Determine the angle between $\vec{u}$ and $\vec{v}$.

Mid-Review: Geometric Vector

SOLUTION

1. The diagram shows a parallelepiped. Determine a single vector (with head and tail on the parallelepiped) that is equivalent to each sum or difference.



a) $-\overrightarrow{AE}+\overrightarrow{DA}=\overrightarrow{\mathbf{EA}}+\overrightarrow{\mathbf{DA}}=\overrightarrow{\mathbf{EA}}+\overrightarrow{\mathbf{HE}}=\overrightarrow{\mathbf{HA}}$

b) $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CG}=\overrightarrow{\mathbf{AB}}+\overrightarrow{\mathbf{BC}}+\overrightarrow{\mathbf{CG}}=\overrightarrow{\mathbf{AG}}$

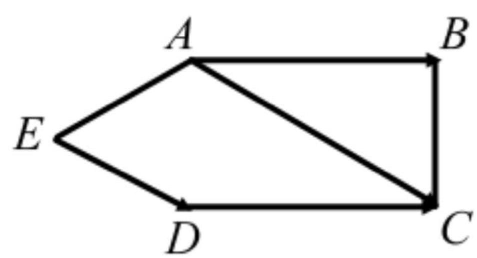
c) $\overrightarrow{FG}-\overrightarrow{DC}-\overrightarrow{AE}+\overrightarrow{AF}=\overrightarrow{\mathbf{FG}}+\overrightarrow{\mathbf{CD}}+\overrightarrow{\mathbf{EA}}+\overrightarrow{\mathbf{AF}}$

$=\overrightarrow{\mathbf{FG}}+\overrightarrow{\mathbf{GH}}+\overrightarrow{\mathbf{EF}}$

$=\overrightarrow{\mathbf{FH}}+\overrightarrow{\mathbf{EF}}$

$=\overrightarrow{\mathbf{EH}}$

2. ABCDE is a pentagon such that $\overrightarrow{AB}=\overrightarrow{DC}$ and $\overrightarrow{AC}=2\overrightarrow{ED}$ .write each vector in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



a) $\overrightarrow{EC}=\overrightarrow{\mathbf{ED}}+\overrightarrow{\mathbf{DC}}=\frac{1}{2}\overrightarrow{\mathbf{AC}}+\overrightarrow{\mathbf{AB}}$

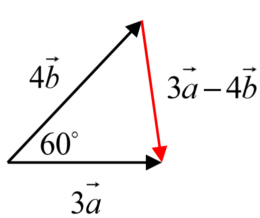
b) $\overrightarrow{CB}=\overrightarrow{\mathbf{AB}}-\overrightarrow{\mathbf{AC}}$

$\overrightarrow{AE}=\overrightarrow{\mathbf{AC}}-\overrightarrow{\mathbf{EC}}$

c) $=\overrightarrow{\mathbf{AC}}-\frac{1}{2}\overrightarrow{\mathbf{AC}}-\overrightarrow{\mathbf{AB}}$

$=\frac{1}{2}\overrightarrow{\mathbf{AC}}-\overrightarrow{\mathbf{AB}}$

3. If $\vec{a}$ and $\vec{b}$ are unit vectors that make an angle of 60^0 with each other, calculate $|3\vec{a}-4\vec{b}|$.



$$|3\vec{a}-4\vec{b}|^2=|3\vec{a}|^2+|4\vec{b}|^2-2(|3\vec{a}|)(|4\vec{b}|)\cos 60^\circ$$

$$=9+16-2(3)(4)\left(\frac{1}{2}\right)$$

$$=13$$

$$|3\vec{a}-4\vec{b}|=\sqrt{13}\text{ unit}$$

Mid-Review: Geometric Vector

4. If $\frac{2}{3}\vec{x}=\vec{a}+\frac{1}{3}\vec{b}$, $2\vec{y}=-3\vec{a}+\vec{b}$ express $2\vec{a}-5\vec{b}$ in terms of $\vec{x},\vec{y}$.

$$2\vec{x}=3\vec{a}+\vec{b}$$
$$\underline{2\vec{y}=-3\vec{a}+\vec{b}}$$
$$2\vec{x}+2\vec{y}=2\vec{b}\rightarrow\vec{b}=\vec{x}+\vec{y}$$
$$2\vec{x}=3\vec{a}+\vec{b}\xrightarrow{\vec{b}=\vec{x}+\vec{y}}2\vec{x}=3\vec{a}+\vec{x}+\vec{y}$$
$$\vec{x}-\vec{y}=3\vec{a}$$
$$\vec{a}=\frac{1}{3}\left(\vec{x}-\vec{y}\right)$$
$$2\vec{a}-5\vec{b}=\frac{2}{3}\left(\vec{x}-\vec{y}\right)-5\left(\vec{x}+\vec{y}\right)$$
$$=\frac{-1}{3}\left(13\vec{x}+17\vec{y}\right)$$

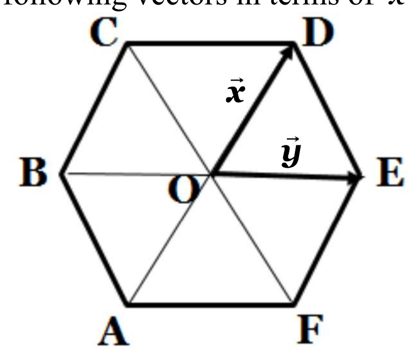
5. Given that

$$\vec{u}=x\vec{a}+2y\vec{b}$$
$$\vec{v}=-2y\vec{a}+3y\vec{b}$$
$$\vec{w}=4\vec{a}-2\vec{b}$$

where $\vec{a}$ and $\vec{b}$ are not collinear, find the values of x and y for which $2\vec{u}-\vec{v}=\vec{w}$.

$$2\left(x\vec{a}+2y\vec{b}\right)-\left(-2y\vec{a}+3y\vec{b}\right)=4\vec{a}-2\vec{b}$$
$$2x\vec{a}+4y\vec{b}+2y\vec{a}-3y\vec{b}=4\vec{a}-2\vec{b}$$
$$\vec{a}\left(2x+2y\right)+\vec{b}y=4\vec{a}-2\vec{b}$$
$$2x+2y=4\xrightarrow{y=-2}\boxed{x=4}$$
$$\boxed{y=-2}$$

6. Using the regular hexagon ABCDEF shown, express each of the following vectors in terms of $\vec{x}$ and $\vec{y}$.



a) $\overrightarrow{DA}=-2\vec{x}$

b) $\overrightarrow{DE}=\vec{y}-\vec{x}$

c) $\overrightarrow{BF}=\overrightarrow{BA}+\overrightarrow{AF}$

$$=\overrightarrow{DE}+\overrightarrow{AF}$$
$$=\vec{y}-\vec{x}+\vec{y}$$
$$=2\vec{y}-\vec{x}$$

Mid-Review: Geometric Vector

7. Given $\vec{p} = [2,-3], \vec{q} = [-1,4]$, evaluate $3\vec{p} - 2\vec{q}$.

$$3\vec{p} - 2\vec{q} = 3[2,-3] - 2[-1,4]$$
$$= [8,-17]$$
$$|3\vec{p} - 2\vec{q}| = \sqrt{8^2 + (-17)^2}$$
$$= \sqrt{353}$$
$$\approx 18.8$$

8. Given the point $P = (4,-3)$ where $\vec{PQ} = [7,-4]$ find

a) coordinates of Q

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$
$$[7,-4] = [x,y] - [4,-3]$$
$$[x,y] = [11,-7]$$

b) $|\vec{PQ}|$

$$|\vec{PQ}| = \sqrt{7^2 + 4^2}$$
$$= \sqrt{65}$$

c) a unit vector in the direction of $\vec{QP}$

$$\hat{a} = \frac{\vec{QP}}{|\vec{QP}|}$$
$$= \frac{1}{\sqrt{65}}[-7,4]$$

9. If $\vec{u}$ =[1,4,-2], $\vec{v} = -2\hat{i} - 3\hat{j}$ and $\vec{w}$ =[-1,-3,1], find:

a) $|3\vec{v} + 3\hat{i} - 2\hat{k}|$ **Ans :** $\sqrt{94}$

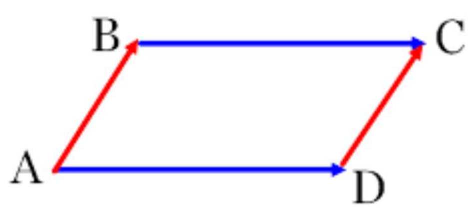
b) a unit vector with the same direction as $\vec{u}$. **Ans :** $\hat{u} = \frac{1}{\sqrt{21}}[1,4,-2]$

c) Find the angle between $\vec{v}$ and $\vec{w}$ **Ans :** $\theta = 23.1°$

*9. a) $|3\vec{v} + 3\hat{i} - 2\hat{k}| = |3(-2\hat{i} - 3\hat{j}) + 3\hat{i} - 2\hat{k}|$
$= |-3\hat{i} + 9\hat{j} - 2\hat{k}|$
$= \sqrt{(-3)^2 + (-9)^2 + (-2)^2}$
$= \sqrt{94}$

10. The points A(-1,2,-1) ,B(2,-1,3), and D(-3,1,-3) are three vertices of parallelogram ABCD. Find the coordinate of C.

$$\vec{AB} = \vec{DC}$$
$$\vec{OB} - \vec{OA} = \vec{OC} - \vec{OD}$$
$$[3,-3,4] = \vec{OC} - [-3,1,-3]$$
$$\vec{OC} = [3,-3,4] + [-3,1,-3]$$
$$\vec{OC} = [0,-2,1]$$
$$C = (0,-2,1)$$



*9. b) $|\vec{u}| = \sqrt{(1)^2 + (4)^2 + (-2)^2}$
$= \sqrt{21}$
$\hat{u} = \frac{[1,4,-2]}{\sqrt{21}}$

*9. c) $\cos\theta = \frac{[-2,-3,0] \cdot [-1,-3,1]}{\sqrt{(-2)^2 + (-3)^2}\sqrt{(-1)^2 + (-3)^2 + (1)^2}}$
$= \frac{2+9}{\sqrt{13}\sqrt{11}}$
$= \frac{11}{\sqrt{143}}$
$\theta = \cos^{-1}\left(\frac{11}{\sqrt{143}}\right)$
$\doteq 23.1°$

Mid-Review: Geometric Vector

11. Vectors [2, -a, 1] and [−2, 2, -a+1] are collinear. Find the value of a.

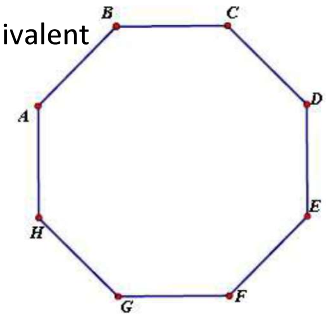
$$[2,-a,1]=k[-2,2,-a+1]$$
$$[2,-a,1]=[-2k,2k,k(-a+1)]$$
$$2=-2k \rightarrow k=-1$$
$$-a=2k$$
$$a=-2(-1)$$
$$\boxed{a=2}$$
$$1=k(-a+1)$$
$$1=a-1$$
$$\boxed{a=2}$$

12. The vectors $\vec{u}$ and $\vec{v}$ have lengths 2 and 1 respectively. The vectors $\vec{u}+5\vec{v}$ and $2\vec{u}-3\vec{v}$ are perpendicular. Determine the angle between $\vec{u}$ and $\vec{v}$.

$$(\vec{u}+5\vec{v})\perp(2\vec{u}-3\vec{v})\Rightarrow(\vec{u}+5\vec{v})\bullet(2\vec{u}-3\vec{v})=0$$
$$2\vec{u}\bullet\vec{u}-3\vec{u}\bullet\vec{v}+10\vec{v}\bullet\vec{u}-15\vec{v}\bullet\vec{v}=0$$
$$2\left|\vec{u}\right|^2+7\left(\left|\vec{u}\right|\right)\left(\left|\vec{v}\right|\right)\cos(\theta)-15\left|\vec{v}\right|^2=0$$
$$2(2)^2+7(2)(1)\cos\theta-15(1)^2=0$$
$$14\cos(\theta)=7$$
$$\cos(\theta)=\frac{1}{2}$$
$$\theta=60^{\circ}$$

Unit 2 – Algebraic Vectors Review

1. The diagram on the right shows a regular octagon. Write a single vector that is equivalent to each vector expression



   a. $\overrightarrow{HA} + \overrightarrow{AB}$ $\qquad [\overrightarrow{HB}]$
   b. $\overrightarrow{GH} - \overrightarrow{GF}$ $\qquad [\overrightarrow{FH}]$
   c. $\overrightarrow{FE} + \overrightarrow{BA}$ $\qquad [\vec{0}]$
   d. $\overrightarrow{GA} - \overrightarrow{EH} + \overrightarrow{DG}$ $\qquad [\vec{0}]$

2. Given $\vec{u} = [-2, y]$ and $\vec{u}$ makes a $120°$ with the x-axis. Determine the value of y and $|\vec{u}|$.
   $[y = 2\sqrt{3},\ |\vec{u}| = 4]$

3. Using vectors show that the three points $A(2,-3,7)$, $B(7,12,-3)$, and $C(-2,-15,15)$ are collinear.

4. Determine the value of k so that $\vec{u} = [k, 3]$ and $\vec{v} = [k, 2k]$ are perpendicular. $\qquad [k = -6]$

5. Solve for x if $\vec{u} = [3x, 7], \vec{v} = [5x, x], and\ |\vec{u} + \vec{v}| = 10x$. $\qquad \left[x = \frac{7}{5}\right]$

6. Given that $\vec{u} = 3\vec{x} - \vec{y}$ and $\vec{v} = 2\vec{x} + 5\vec{y}$
   a. Express $\vec{w} = \vec{u} + \vec{v} - 2\vec{x} + \vec{y}$ in terms of $\vec{x}$ and $\vec{y}$. $\qquad [\vec{w} = 3\vec{x} + 5\vec{y}]$
   b. If $\vec{x} = [1, 3]$ and $\vec{y} = [-2, 5]$ then
      i. determine $|\vec{w}|$. $\qquad [\sqrt{1205}]$
      ii. determine the angle $\vec{w}$ makes with the x-axis. $\qquad [101.6°]$

7. Given that $|\vec{a}| = 10$, $|\vec{b}| = 15$, and $|\vec{a} - \vec{b}| = 11$
   a. find the angle between $\vec{a}$ and $\vec{b}$. $\qquad [47.156°]$
   b. calculate $|\vec{a} + \vec{b}|$. $\qquad [23]$

8. In a quadrilateral ABCD, T is the midpoint of the side AB. U is the midpoint of the side CD. L is the midpoint of the diagonal AC, and M is the midpoint of the diagonal BD. Let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$ and $\overrightarrow{CD} = \vec{c}$.
   a. Show that $\overrightarrow{AD} + \overrightarrow{BC} = 2\overrightarrow{TU}$.
   b. Show that $\overrightarrow{AD} + \overrightarrow{CB} = 2\overrightarrow{LM}$.

9. a. If $\cos(\theta) = X$ and θ is obtuse, what do you know about the sign of X?
   b. If the angle between $\vec{u}$ and $\vec{v}$ is obtuse, what do you know about the value of $\vec{u} \cdot \vec{v}$?
   c. Determine the value(s) of k so that the angle between $\vec{x} = [11, 3, 2k]$ and $\vec{y} = [k, 4, k]$ is obtuse.
   $\left[-4 < k < -\frac{3}{2}\right]$

10. If $\vec{u} = [1, 4, -2], \vec{v} = -2\hat{\imath} - 3\hat{\jmath}, and\ \vec{w} = [-1, -3, 1]$ find
    a. $|3\vec{v} + 3\hat{\imath} - 2\hat{k}|$. $\qquad [\sqrt{94}]$
    b. $\hat{u}$. $\qquad \left[\frac{1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, -\frac{2}{\sqrt{21}}\right]$
    c. the angle between $\vec{v}\ and\ \vec{w}$. $\qquad [23.09°]$

d. a vector with a magnitude of 7 in the opposite direction of $\vec{w}$. $\left[\frac{7}{\sqrt{11}}[1,3,-1]\right]$

11. The points $A(-1,2,-1)$, $B(2,-1,3)$ and $D(-3,1,-3)$ are three vertices of parallelogram ABCD.
    a. Find the coordinate of C . $[C(0,-2,1)]$
    b. Verify that the vector $-10\hat{\imath}+2\hat{\jmath}+9\hat{k}$ is perpendicular to $\overrightarrow{AB}$.

12. If the points $A(1,-1,4)$, $B(1,1,2)$, and $C(2,-1,-1)$ are the vertices of a triangle, determine $\overrightarrow{BA}$, $\overrightarrow{BC}$, and $\angle ABC$. Use these to determine the area of ΔABC. $\left[\frac{\sqrt{14}\times\sqrt{8}\sin(79.1°)}{2}\right]$

13. If $\vec{a}$ and $\vec{b}$ are unit vectors, and $|\vec{a}+\vec{b}|=\sqrt{3}$ determine $(2\vec{a}-5\vec{b})\cdot(\vec{b}+3\vec{a})$. $\left[-\frac{11}{2}\right]$

14. Determine the vector and parametric equation of a line which passes through the point $A(-4,0,3)$ and is parallel to the x-axis.

15. Determine the symmetric equation of a line which passes through the points $A(2,3,-1)$ and $B(5,-2,9)$. $\left[\frac{x-2}{3}=\frac{y-3}{-5}=\frac{z+1}{10}\right]$

16. Determine the Cartesian/Scalar equation of a line which goes through the point $(-3,5)$ that is normal to the line $y=\frac{2}{3}x-7$. $[3x+2y-1=0]$

17. Determine the vector equation of a line which goes through the point $(3,4)$ that is
    a. parallel to the line $y=-\frac{4}{3}x+1$. $[l:[x,y]=(3,4)+t[-3,4]]$
    b. perpendicular to the line $y=2x+5$. $[l:[x,y]=(3,4)+t[2,-1]]$

18. Write the parametric equation of the line that goes through the point $(6,-2,1)$ and is perpendicular to both

$$l_1:[x,y,z]=[1,4,-2]+t[3,-1,1]$$
$$l_2:[x,y,z]=[9,5,-3]+k[1,-3,7]$$

$[x=6-t, y=-2-5t, z=1-2t]$

19. Which of these vector equations represent the same line? $[l_1 \; and \; l_3]$

$$l_1:[x,y,z]=[11,-2,17]+t[3,-1,4]$$
$$l_2:[x,y,z]=[-13,6,-10]+k[-3,1,-4]$$
$$l_3:[x,y,z]=[-7,4,-7]+s[-6,2,-8]$$

20. Given the lines $l_1:[x,y,z]=(3,-7,5)+k[1,-2,4]$ and $l_2: [x,y,z]=(-7,-8,0)+m[3,1,-1]$. Determine if the lines intersect. If the lines intersect state the intersection point and determine the acute angle between both lines. $[skew \; lines]$

21. Given lines $l_1:\frac{x-3}{1}=\frac{y+7}{-2}=\frac{5-z}{-4}$ and $l_2:\frac{x+7}{3}=\frac{y+8}{1}=\frac{-z+4}{1}$. Determine if the lines intersect. If the lines intersect state the intersection point and determine the acute angle between both lines. $[(2,-5,1), \; 78.62°]$

22. Does the line $l_1: [x, y, z] = [-4, 2, -2] + t[2, -1, 3]$
    a. Intersect the z-axis? If so, where? $[0, 0, 4]$
    b. Intersect the y-axis? If so, where? $[no\ intersection]$

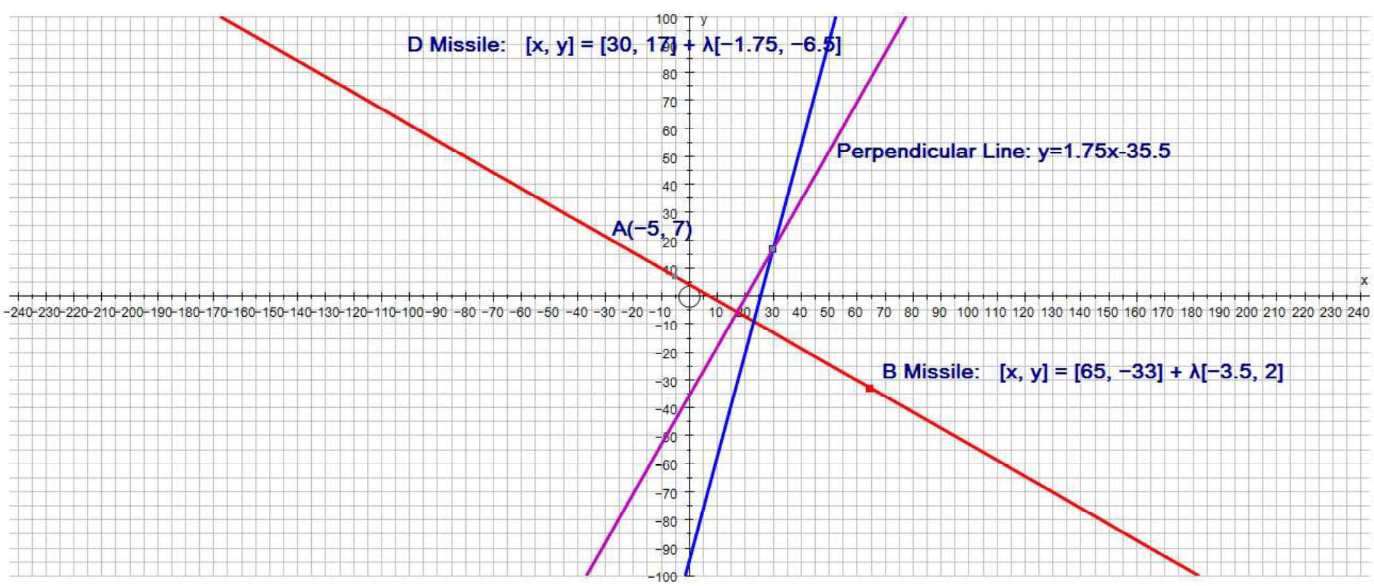
23. The position of two helicopters X and Y at time t seconds are given by the formula

$$H_1: [x, y, z] = (11, 3, -3) + t[1, -1, 4]$$
$$H_2: [x, y, z] = (1, -7, -2) + s[2, 1, 9]$$

    a. What is the speed of the two helicopters if distances are measured in metres? $\left[3\sqrt{2}\ and\ \sqrt{86}\right]$
    b. Show that the two helicopters will not collide. $\left[s = \frac{20}{3}, t = \frac{10}{3}\ x\ and\ y\ coordinaes\right]$
    c. Determine the distance between the helicopters when $t = 10$. $\left[\sqrt{2701}\ m\right]$

24. An Enemy Battleship is located at point $B(65, -33)$ notices a stranded Aircraft Carrier located at point $A(-5, 7)$. The Battleship fires a missile towards the Carrier with a velocity of $\vec{v} = [-3.5, 2]\ units/min$. If a Friendly Destroyer located at $D(30, 17)$ notices the missile on sonar 5 minutes after the missile was launched and is able to fire a counter missile with a velocity of $\vec{v} =$ [-1.75, -6.5] $units/min$, how much time do they have before they must fire their counter missile? $[3\ min]$

25. In Question 24, if the Destroyer wanted to have the most accuracy with a missile, they would want to hit the enemy missile at the time when it is closest to the Destroyer.
    a. Determine the time after the enemy missile is shot when the enemy missile is closest to the Destroyer. $[13.6923\ min]$
    b. What is the coordinate of the point of impact? $[(17.07695, -5.6154)]$



Unit 2 - Algebraic Vectors Review

1. b) $\vec{GH} - \vec{GF}$
$= \vec{FG} + \vec{GH}$
$= \vec{FH}$

c) $\vec{FE} + \vec{BA}$
$= \vec{FE} + \vec{EF}$
$= \vec{0}$

d) $\vec{GA} - \vec{EH} + \vec{DG}$
$= \vec{DG} + \vec{GA} - \vec{EH}$
$= \vec{DA} - \vec{EH}$
$= \vec{DA} - \vec{DA}$
$= \vec{0}$

2. $\hat{i} = [1,0]$

$$\cos 120^\circ = \frac{[-2, y]\cdot[1,0]}{\sqrt{4+y^2}\cdot\sqrt{1}}$$

$$\sqrt{4+y^2} = \frac{-2}{-0.5}$$

$4 + y^2 = 16$
$y^2 = 12$
$y = \pm 2\sqrt{3}$
inadmissable

$|\vec{u}| = \sqrt{(-2)^2 + (\sqrt{12})^2}$
$= \sqrt{4+12}$
$= 4u$.

3. $\vec{AB} = [5, 15, -10]$ $\quad \vec{AC} = [-4, -12, 8]$.

$5t = -4$
$t = -\frac{4}{5}$

$\cancel{15}^{3}(-\frac{4}{\cancel{5}})$
$= -12$

$-\cancel{10}^{2}(-\frac{4}{\cancel{5}})$
$= 8$

$\therefore -\frac{4}{5}\vec{AB} = \vec{AC}$
$\therefore$ A, B & C are collinear

4. $[k, 3]\cdot[k, 2k] = 0$
$k^2 + 6k = 0$
$\cancel{k=0}$ or $k = -6$

5. $\vec{u}+\vec{v} = [8x, 7+x]$

$$|\vec{u}+\vec{v}| = \sqrt{(8x)^2+(7+x)^2}$$

$$10x = \sqrt{64x^2+49+14x+x^2}$$

$$100x^2 = 65x^2+14x+49$$

$$35x^2-14x-49=0$$

$$(7x+7)(5x-7)=0$$

$$\cancel{x=-1} \text{ or } x=\frac{7}{5}$$

inadmissable since magnitude cannot be negative

6. a) $\vec{w} = 3\vec{x}-\vec{y}+2\vec{x}+5\vec{y}-2\vec{x}+\vec{y}$

$= 3\vec{x}+5\vec{y}$

b) i) $\vec{w} = 3[1,3]+5[-2,5]$

$= [3,9]+[-10,25]$

$= [-7,34]$

$|\vec{w}| = \sqrt{(-7)^2+(34)^2}$

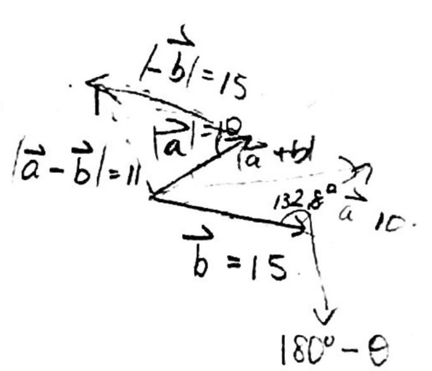
$= \sqrt{1205}$ u.

ii) $\cos\theta = \dfrac{[-7,34]\cdot[1,0]}{\sqrt{1205}\sqrt{1}}$

$\theta = \cos^{-1}\left(\dfrac{-7}{\sqrt{1205}}\right)$

$\doteq 101.6^\circ$

7.



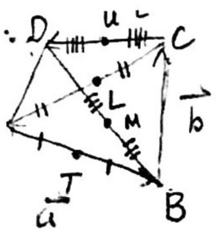
$11^2 = 10^2+15^2-2(10)(15)\cos\theta$

$\theta \doteq 47.2^\circ$

$|\vec{a}+\vec{b}|^2 \doteq 10^2+15^2-2(10)(15)\cos 132.8^\circ$

$|\vec{a}+\vec{b}| \doteq 23u$

8.



a) $\vec{AD} + \vec{BC}$

$= \vec{AD} + \vec{BA} + \vec{AD} + \vec{DC}$

$= 2\vec{AD} + 2\vec{TA} + 2\vec{Du}$

$= 2(\vec{TA} + \vec{AD} + \vec{Du})$

$= 2\vec{Tu}$

b) $\vec{AD} + \vec{CB}$

$= \vec{AL} + \vec{LD} + \vec{CA} + \vec{AB}$

$= \vec{AL} + \vec{LM} + \cancel{\vec{MD}} + 2\vec{LA} + \vec{AM} + \cancel{\vec{MB}}$ $\quad \vec{MB} = -\vec{MD}$

$= \vec{AM} + 2\vec{LA} + \vec{AM}$

$= 2\vec{AM} + 2\vec{LA}$

$= 2(\vec{LA} + \vec{AM})$

$= 2\vec{LM}$

RS

$2\vec{LM}$

$= 2(\vec{LA} + \vec{AM})$

#8b) $\vec{AD} + \vec{CB}$

$= \vec{AB} + \vec{BC} + \vec{CD} + \vec{CB}$

$= \vec{a} + \vec{b} + \vec{c} - \vec{b}$

$= \vec{a} + \vec{c}$

$2\vec{LM} = 2(\vec{LC} + \vec{CD} + \vec{DM})$

$= 2(\frac{1}{2}\vec{AC} + \vec{c} + \frac{1}{2}\vec{DB})$

$= \vec{AC} + 2\vec{c} + \vec{DB}$

$= \vec{a} + \vec{b} + 2\vec{c} + (-\vec{c} - \vec{b})$

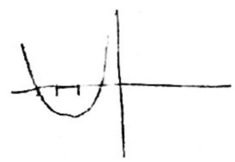
$= \vec{a} + \vec{c}$

9. a) -ve, since Q2

b) $\vec{u} \cdot \vec{v} = -ve$ $\therefore \theta$ is -ve, $\vec{x} \cdot \vec{y} = -ve$

c) $[11, 3, 2k] \cdot [k, 4, k] < 0$

$11k + 12 + 2k^2 < 0$



$2k^2 + 11k + 12 < 0$

$(2k+3)(k+4) < 0$ $\quad \therefore -4 < k < -\frac{3}{2}$

$k = -\frac{3}{2}$ or $k = -4$

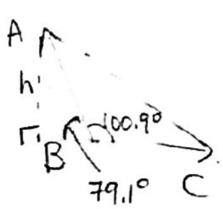
12. $\vec{BA} = [0, -2, 2]$ $\quad \cos\theta = \dfrac{0+4-6}{\sqrt{8}\sqrt{14}}$

$\vec{BC} = [1, -2, -3]$ $\quad \theta = 100.9°$

$|\vec{BA}| = \sqrt{4+4}$ $\quad \vec{BC} = \sqrt{14}$

$= \sqrt{8}$

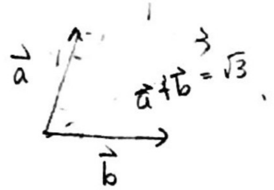
$A = \frac{1}{2}\sqrt{14} \cdot \sqrt{8} \sin 79.1°$



$\sin\theta = \dfrac{h}{\sqrt{8}}$

$h = \sqrt{8} \sin 79.1°$

13.



$(2\vec{a} - 5\vec{b})\cdot(\vec{b} + 3\vec{a})$

$= 2\vec{a}\cdot\vec{b} + 6|\vec{a}|^2 - 5|\vec{b}|^2 - 15\vec{a}\cdot\vec{b}$

$= 1 - 13\vec{a}\cdot\vec{b}$

$(\sqrt{3})^2 = 1^2 + 1^2 - 2(1)(1)\cos\theta$

$= 1 - 13|\vec{a}||\vec{b}|\cos\theta$

$\theta = \cos^{-1}\left(\frac{-1}{2}\right)$

$= 1 - 13\cos 120^\circ$

$= 120^\circ$

$= -5.5$

$A(-4, 0, 3)$ $(1, 0, 0)$.

14. VE: $\vec{d} = [5, 0, -3]$

$\vec{r} = [-4, 0, 3] + t[5, 0, -3]$

$x = -4 + 5t$

$y = 0$

$z = 3 - 3t$

15. $\vec{d} = [3, -5, 10]$

$x = 2 + 3t$

$y = 3 - 5t$

$z = -1 + 10t$

$\frac{x-2}{3} = \frac{y-3}{-5} = \frac{z+1}{10}$

16. $m = \frac{2}{3}$

$m_\perp = -\frac{3}{2}$

$m = -\frac{A}{B}$

$\therefore A = 3, B = 2$

$3x + 2y + C = 0$

$3(-3) + 2(5) + C = 0$

$-9 + 10 + C = 0$

$C = -1$

$\therefore 3x + 2y - 1 = 0$

17. a) $m = -\frac{4}{3}$ $\quad \vec{d} = [3,-4]$ or $[-3,4]$.

$\vec{r} = [3,4] + t[3,-4]$.

b) $m = 2$ $\quad \vec{d} = [2,-1]$ or $[-2,1]$.

$m_b = -\frac{1}{2}$.

$\vec{r} = [3,4] + t[2,-1]$.

18. skip

19. $\vec{d}_1 = [3,-1,4]$ $\quad \vec{d}_2 = [-3,1,-4]$ $\quad \vec{d}_3 = [-6,2,-8]$.

$\vec{d}_1 = -\vec{d}_2$ $\quad \vec{d}_1 = -2\vec{d}_3$ $\quad \vec{d}_2 = 2\vec{d}_3$

sub $(11,-2,17)$ into l_2:

$11 = -13-3k$ $\quad -2 = 6+k$ $\quad 17 = -10-4k$

$24 = -3k$ $\quad -8 = k$ $\quad 27 = -4k$

$k = -8$ $\quad k = \frac{27}{-4}$

$\therefore$ k values are not the same, l_1 & l_2 are distinct.

sub $(11,-2,17)$ into l_3:

$11 = -7-6s$ $\quad -2 = 4+2s$ $\quad 17 = -7-8s$

$18 = -6s$ $\quad -6 = 2s$ $\quad 24 = -8s$

$s = -3$ $\quad s = -3$ $\quad s = -3$

$\therefore$ k values are the same, l_1 & l_2 are the same

sub $(-13,6,-10)$ into l_3:

$-13 = -7-6s$ $\quad 6 = 4+2s$ $\quad -10 = -7-8s$

$-6 = -6s$ $\quad 2 = 2s$ $\quad -3 = -8s$

$s = 1$ $\quad s = 1$ $\quad s = \frac{3}{8}$

not the same line

20. $\vec{d}_1 = [1,-2,4]$ $\vec{d}_2 = [3,1,-1]$

$\because \vec{d}_1 \neq k_1\vec{d}_2$, $k \in \mathbb{R}$

$\therefore$ lines are not //

$3+k = -7+3m$

$k = -10+3m$

$k = -10+3(3)$

$= -1$

$-7-2k = -8+m$

$-7-2(-10+3m) = -8+m$

$-7+20-6m = -8+m$

$7m = 21$

$m = 3$

$5+4k = -m$

Check: LS $5+4(-1) = 1$ RS $-(3) = -3$

$\because$ LS $\neq$ RS

$\therefore$ lines are skew.

21. $\vec{d}_1 = [1,-2,4]$ $\vec{d}_2 = [3,1,-1]$

$\because \vec{d}_1 \neq k\vec{d}_2$

$\therefore$ lines are not //

$t+3 = 3k-7$

$t = 3k-10$

$t = -1$

$-2t-7 = k-8$

$-2(3k-10)-7 = k-8$

$-6k+20-7 = k-8$

$7k = 21$

$k = 3$

$4t+5 = -k+4$

LS $4(-1)+5 = 1$ RS $-(3)+4 = 1$

$\because$ LS = RS

$\therefore$ lines will intersect

$x = 2$
$y = -5$
$z = 1$

$\therefore$ intersection pt is $(2,-5,1)$.

$$\cos\theta = \frac{[1,-2,4]\cdot[3,1,-1]}{\sqrt{21}\sqrt{11}}$$

$$\theta = \cos^{-1}\left(\frac{-3}{\sqrt{231}}\right)$$

$= 101.38^\circ$

Acute $= 78.62^\circ$

22. a) $\vec{d_1} = [2,-1,3]$ $\quad \vec{d_2} = [0,0,1]$

$\because \vec{d_1} \neq k\vec{d_2}$

$\therefore$ z-axis & l are not $\parallel$

$l_1 : \vec{r} = [-4,2,-2] + t[2,-1,3]$ $\quad l_2 : \vec{r} = [0,0,0] + s[0,0,1]$

$-4+2t=0$ $\quad 2-t=0$ $\quad -2+3t=s$

$2t=4$ $\quad t=2$ $\quad -2+3(2)=s$

$t=2$ $\quad 4=s$

$x=0$

$y=0$

$z=4$

b) $l_2 : \vec{r} = [0,0,0] + m[0,1,0]$.

$-4+2t=0$ $\quad 2-t=1$ $\quad -2+3t=0$

$t=2$ $\quad t=1$

$\because$ t values are not the same,

$\therefore$ lines do not intersect.

23. a) $s_1 = \sqrt{1^2+(-1)^2+(4)^2}$ $\quad s_2 = \sqrt{2^2+1^2+9^2}$

$= \sqrt{18}$ $\quad = \sqrt{86}$

$= 3\sqrt{2}$

$\vec{d_1} = [1,-1,4]$ $\quad \vec{d_2} = [2,1,9]$.

b) $11+t = 1+2s$ $\quad 3-t = -7+s$ $\quad -3+4t = -2+9s$

$t = -10+2s$ $\quad 3-(-10+2s) = -7+s$

$t = -10+2\left(\frac{20}{3}\right)$ $\quad 3+10-2s = -7+s$

$= -10+\frac{40}{3}$ $\quad 3s = 20$

$= \frac{10}{3}$ $\quad s = \frac{20}{3}$

LS $\quad$ RS

$-3+4\left(\frac{10}{3}\right)$ $\quad -2+\cancel{9}^{3}\left(\frac{20}{\cancel{3}}\right)$

$= -3+\frac{40}{3}$ $\quad = -2+60$

$= \frac{31}{3}$ $\quad = 58$

$\because LS \neq RS$

$\therefore$ will not collide

23. c) $t=10$

H_1: $x = 11 + (10)1 = 21$ $y = 3 + (10)(-1) = -7$ $z = -3 + (10)4 = 37$ $(21, -7, 37)$

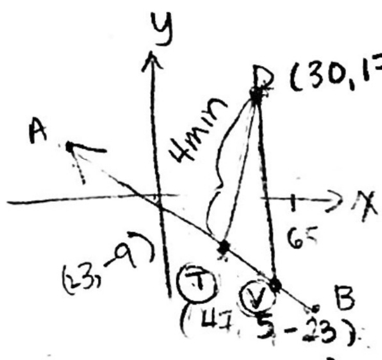
H_2: $x = 1 + (10)2 = 21$ $y = -7 + (10)1 = 3$ $z = -2 + (10)9 = 88$ $(21, 3, 88)$

$\vec{d} = [0, 10, 51]$ $|\vec{d}| = \sqrt{10^2 + 51^2} = \sqrt{2701}$

24. Battleship $B(65, -33)$ $\longrightarrow \vec{r}_{BA} = [65, -33] + t_1[-3.5, 2]$

Aircraft Carrier $A(-5, 7)$

Friendly Destroyer $D(30, 17)$ $\xrightarrow{t=5} \vec{r} = [30, 17] + t_2[-1.75, -6.5]$



Point V: after 5 mins $x = 65 - (3.5)(5) = 47.5$ $y = -33 + (2)(5) = -23$

Intersection between missiles

$x = 65 - 3.5t_1$ $x = 30 - 1.75t_2$

$y = -33 + 2t_1$ $y = 17 - 6.5t_2$

$\vec{TV} = [24.5, -14]$

$|\vec{TV}| = \sqrt{24.5^2 + (-14)^2} = 28.2$ $|\vec{v}| = \sqrt{(-3.5)^2 + (2)^2} \doteq 4.03$

① $65 - 3.5t_1 = 30 - 1.75t_2$

② $-33 + 2t_1 = 17 - 6.5t_2$

① $-3.5t_1 = -35 - 1.75t_2$

$t_1 = 10 + 0.5t_2 = 12$

② $-33 + 2(10 + 0.5t_2) = 17 - 6.5t_2$

$-33 + 20 + t_2 = 17 - 6.5t_2$

$7.5t_2 = 30$

$t_2 = 4$ to get to intersection

Battleship missile $t = \dfrac{d}{v} = \dfrac{28.2}{4.03}$

$t \doteq 7.00$ mins. to get to intersection

∴ it will take the missile 4mins to travel ∴ they have 3mins to launch a counter missile

T: Intersection pt

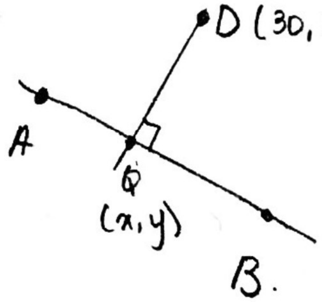
$x = 23$

$y = -9$

$(23, -9)$

a) closest to destroyer when $\overrightarrow{AB} \perp \overrightarrow{DQ}$ ← $\vec{d} = [-7,-26]$

$\vec{r} = [30,17] + t[-7,-26]$



$\vec{r} = [65,-33] + t[-3.5,2]$

$\overrightarrow{DQ} \cdot \overrightarrow{QB} = 0 \qquad \overset{\overrightarrow{AB}}{\overrightarrow{QB}} = [-3.5,2]$

$[30-x, 17-y] \cdot [-3.5,2] = 0$

$-3.5(30-x) + 2(17-y) = 0$

note: pt. Q lies on the line $\overrightarrow{AB}$.

$-3.5[30-(65-3.5t)] + 2[17-(-33+2t)] = 0$

$-3.5(-35+3.5t) + 2(50-2t) = 0$

$122.5 - 12.25t + 100 - 4t = 0$

$-16.25t = -222.5$

$t \doteq 13.7$ mins

b) $x = 65 - 3.5(13.7)$ $\qquad y = -33 + 2(13.7)$

$= 17.1 \qquad = -5.61$

∴ the coordinate of the pt. of impact is $(17.1, -5.61)$.