Unit 2
Probability
&
Probability Distributions

2.1 Sets & Venn Diagrams

A **set** is a collection of things.
For example, the items you wear is a set: these would include shoes, socks, shirt, pants, etc.

The set of PRIME NUMBERS: {2,3,5,7,11,13,... }
The set of WHOLE NUMBERS: {0,1,2,3,... }

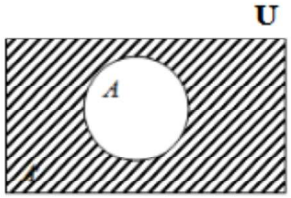
Set Theory Symbols and Definitions

Symbol	Name	Definition	Example
$\{\ \}$	Set	A collection of elements (objects)	$A = \{2,7,8,9,15,23,35\}$
U	Universal set	set that contains all the elements under discussion for a particular situation	
$A \cap B$	Intersection	Objects that belong to set A **and** set B	If set $A = \{1,2,3\}$ and set $B = \{2,3,4\}$, then $A \cap B = \{2,3\}$
$A \cup B$	Union	Objects that belong to set A **or** set B	If set $A = \{1,2,3\}$ and set $B = \{4,5,6\}$, then $A \cup B = \{1,2,3,4,5,6\}$
$A \subseteq B$	Subset	Set A is a subset of set B iff every element of set A is in set B	If set $A = \{a,b,c\}$ and set $B = \{a,b,c,d,e\}$, then $A \subseteq B$
A^c or A'	Complement	All objects that do not belong to set A	If the universal set $U = \{a,b,c,d,e\}$ and $A = \{b,e\}$ then $A' = \{a,c,d\}$

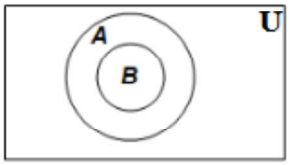
Number of elements in set A = n (A).
Null set empty set - has no objects = Ø or { }.
Disjoint sets - have no common elements. (Also referred to as distinct.)

Venn Diagrams are a way of representing sets.
A Venn Diagram usually consists of a rectangle which represents the sample space, and circles within it represent particular events.

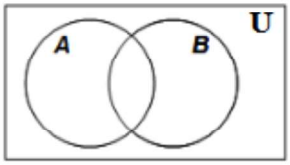
When we have 2 (or more) events we can represent the relationships between the events with the aid of a Venn diagram. The following Venn diagrams show the relationships possible between two events.



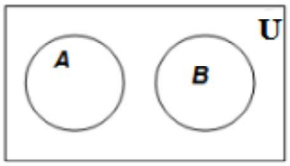
A' is the complement of A



B is a subset of A
All elements of B are in A



A and B have elements in common



A and B have no common elements
A and B are **disjoint sets**
A and B are **mutually exclusive events**

The _____of 2 sets ($A \cup B$) contains the elements that are in A or B or both (inclusive or).

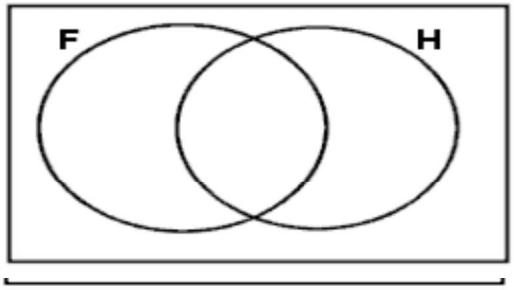
The _____of 2 sets ($A \cap B$) contains the elements that are in A and B.

There is also an exclusive or where A or B means in A or in B but not in both.

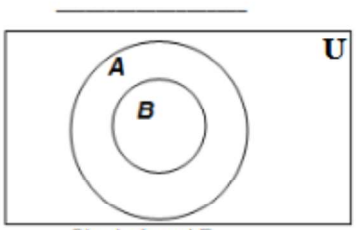
Example 1: In a class there are:

- 8 students who play football and hockey
- 7 students who do not play football or hockey
- 13 students who play hockey
- 19 students who play football

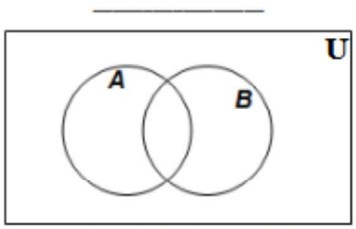
How many students are there in the class?



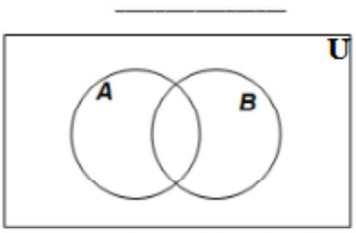
Example 2: Practice shading appropriately and write the proper notation for each scenario.



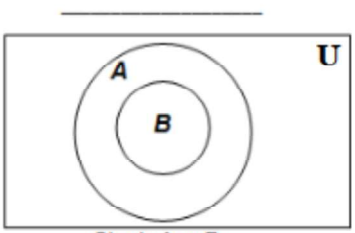
Shade A and B



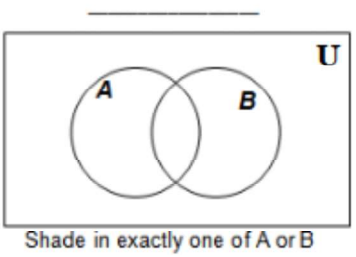
Shade in A but not in B



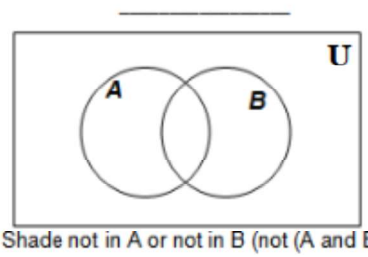
Shade not in A and not in B (neither A nor B)



Shade A or B



Shade in exactly one of A or B



Shade not in A or not in B (not (A and B))

Example 3: Consider the following sample space: S ={1, 2,3, 4,5, 6,7, 8, 9,10} and the event A ={ 1, 2,3, 4,5} , B = {2,3, 4} , C = {6,7} , D = {3, 4,7, 8} . List the elements in the following compound events:

a) $A \cup B$

b) $A \cup D$

c) B'

d) $A \cap B$

e) $A \cap C$

f) $A \cap D'$

g) $A' \cup D'$

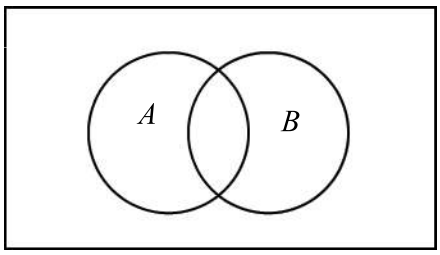
h) $(A \cap D)'$

i) $A' \cap D'$

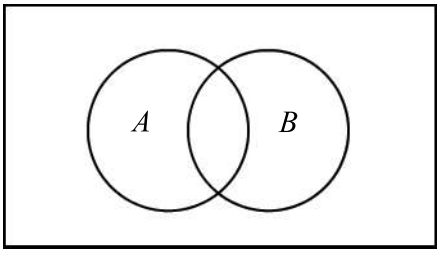
j) $(A \cup D)'$

Example 4: De Morgan's Laws: Given two sets, $A, B \subset U$, $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$
Shade in the Venn Diagrams for:

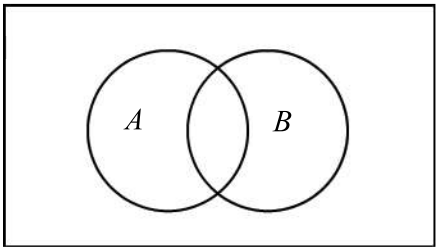
$A' \cup B'$



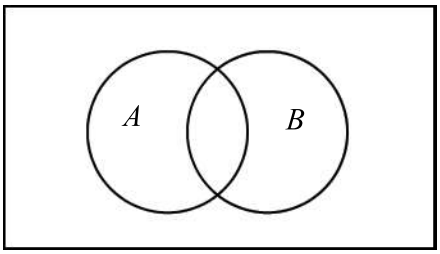
$(A \cap B)'$



$A' \cap B'$



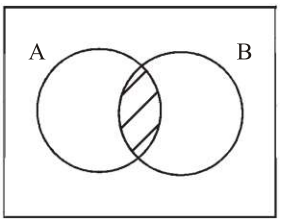
$(A \cup B)'$



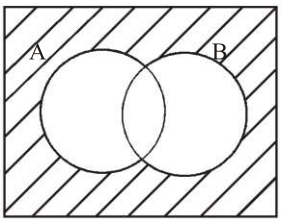
2.1 Practice

1. Write down an expression to describe the shaded area on the following Venn diagrams:

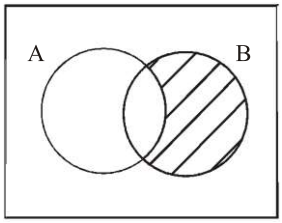
(a) U



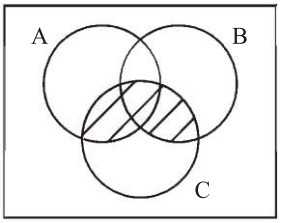
(b) U



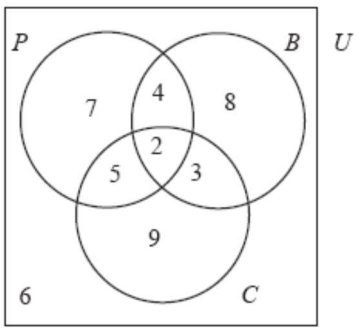
(c) U



(d) U



2. The Venn diagram shows the numbers of pupils in a school according to whether they study the sciences Physics (P), Chemistry (C), Biology (B).



(a) Write down the number of pupils that study Chemistry only.

(b) Write down the number of pupils that study **exactly** two sciences.

(c) Write down the number of pupils that do not study Physics.

(d) Find $n[(P \cup B) \cap C]$.

3. Let

M = {positive integers less than 15};

X={multiple of 2};

Y ={multiples of 3}.

(a) Show, in a Venn diagram, the relationship between the **sets M**, X and Y.

(b) List the elements of:

(i) $X \cap Y$

(ii) $X \cap \neg Y$.

(c) Find the **number of elements** in the complement of $(X \cup Y)$

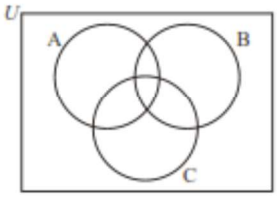
4. The following results were obtained from a survey concerning the reading habits of students.

60% read magazine P
50% read magazine Q
50% read magazine R
30% read magazines P and Q
20% read magazines Q and R
30% read magazines P and R
10% read all three magazines

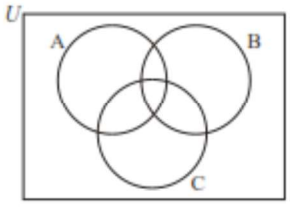
Represent all of this information on a Venn diagram.

5. Shade the given region on the corresponding Venn Diagram.

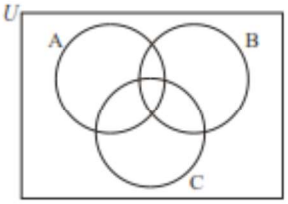
(a) $A \cap B$



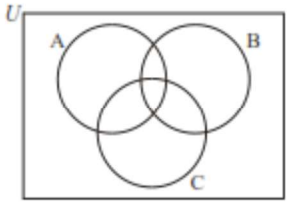
(b) $C \cup B$



(c) $(A \cup B \cup C)'$

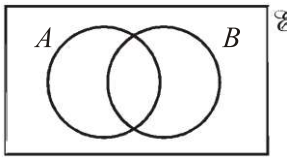


(d) $A \cap C'$

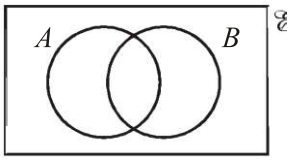


6. In each of the Venn diagrams, shade the region indicated.

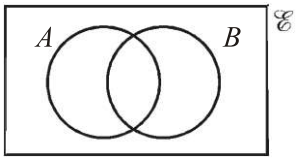
(a) $A \cap B$



(b) The complement of $(A \cap B)$

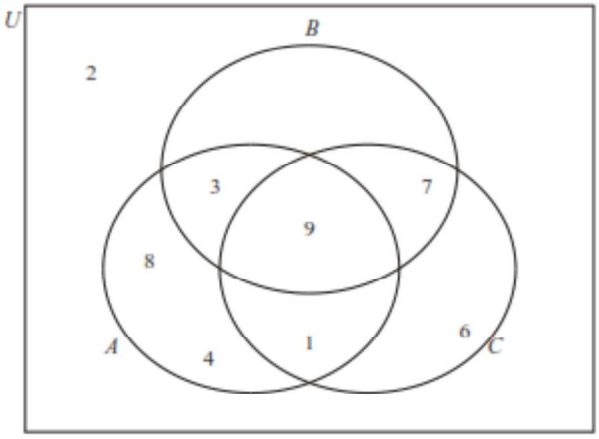


(c) The complement of $(A \cup B)$



7. In the Venn diagram below, A, B and C are subsets of a universal set $U = \{1,2,3,4,6,7,8,9\}$.
List the elements in each of the following sets.

(a) $A \cup B$
(b) $A \cap B \cap C$
(c) $(A' \cap C) \cup B$



8. A group of 30 children are surveyed to find out which of the three sports cricket (C), basketball (B) or volleyball (V) they play. The results are as follows:

3 children do not play any of these sports
2 children play all three sports
6 play volleyball and basketball
3 play cricket and basketball
6 play cricket and volleyball
16 play basketball
12 play volleyball.

(a) Draw a Venn diagram to illustrate the relationship between the three sports played.

(b) On your Venn diagram indicate the number of children that belong to each region.

(c) How many children play only cricket?

9. The universal set U is defined as the set of positive integers less than 10. The subsets A and B are defined as:

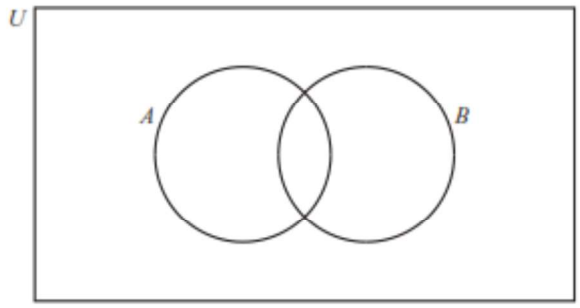
A = {integers that are multiples of 3}
B = {integers that are factors of 30}

(a) List the elements of

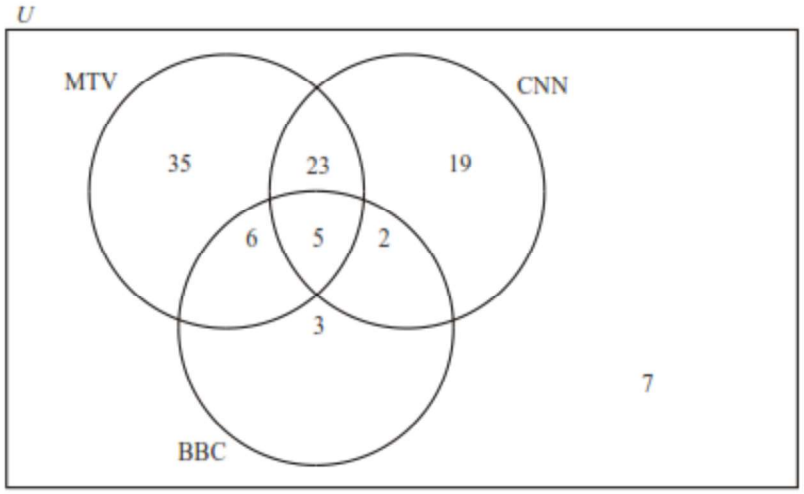
(i) A;

(ii) B.

(b) Place the elements of A and B in the appropriate region in the Venn diagram below.



10. 100 students were asked which television channel (MTV, CNN or BBC) they had watched the previous evening. The results are shown in the Venn diagram below.



From the information in the Venn diagram, write down the number of students who watched

(a) both MTV and BBC;

(b) MTV or BBC;

(c) CNN and BBC but not MTV;

(d) MTV or CNN but not BBC.

2.1 Practice- Answers

1. (a) $A \cap B$

(b) $(A \cup B)'$ or $A' \cap B'$

(c) $A' \cap B$

(d) $(A \cup B) \cap C$ or $(A \cap C) \cup (B \cap C)$

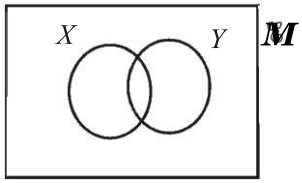
2. (a) 9

(b) 12

(c) $8 + 3 + 9 + 6 = 26$

(d) $5 + 2 + 3$
$= 10$

3. (a)

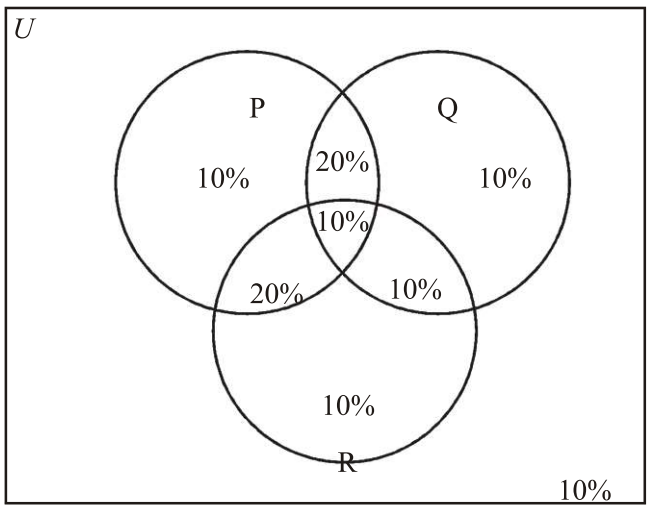


(b) (i) $(X \cap Y) = \{6, 12\}$

(ii) $X \cap \quad Y = \{2, 4, 8, 10, 14\}$

(c) $(X \cup Y)' = \quad (X \cup Y) = \{1, 5, 7, 11, 13\}$
$n(X \cup Y)' = 5$

4. (a)

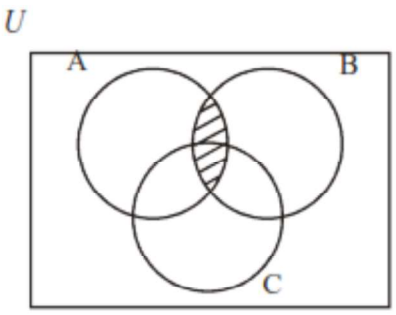


(b) 50% read exactly two magazines

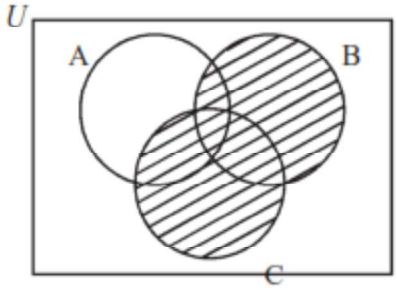
(c) 60% read at least two magazines

(d) 10% do not read any magazines

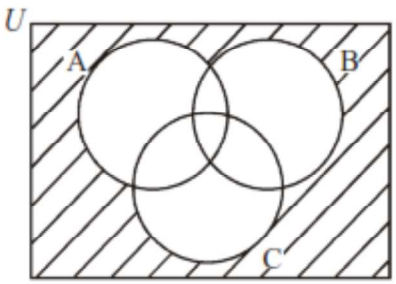
5. (a)



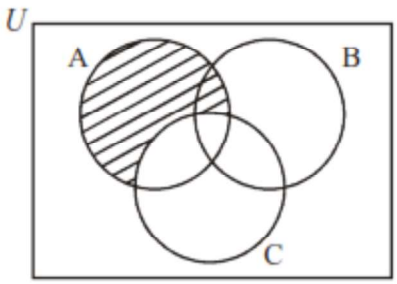
(b)



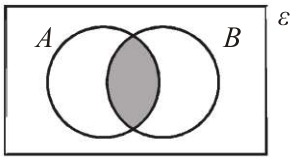
(c)



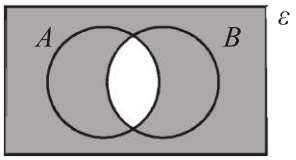
(d)



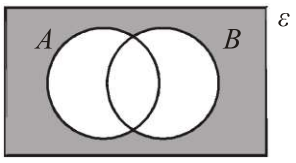
6. (a) $A \cap B$



(b) The complement of $(A \cap B)$

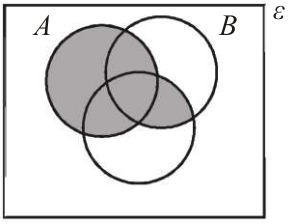


(c) The complement of $(A \cup B)$



C

(d) $A \cup (B \cap C)$



7.

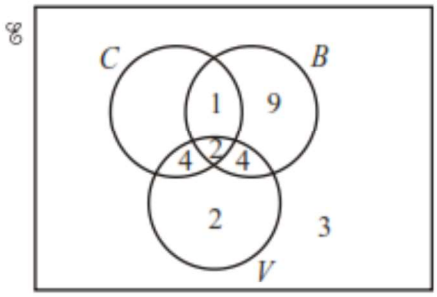
(a) $A \cup B = \{1, 3, 4, 7, 8, 9\}$

(b) $A \cap B \cap C = \{9\}$

(c) $A' = \{1, 3, 4, 7, 8, 9\}$
$A' \cap C = \{6, 7\}$
$(A' \cap C) \cup B = \{3, 6, 7, 9\}$

8. (a)

(b)

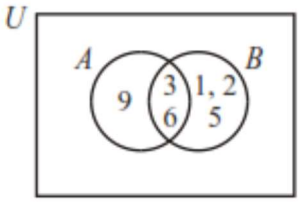


(c) $1 + 9 + 4 + 2 + 4 + 2 + 3 = 25$
$n(C) = 30 - 25$
$= 5$

9. (a) (i) $A = \{3, 6, 9\}$

(ii) $B = \{1, 2, 3, 5, 6\}$

(b)



10.

(a) $n(\text{MTV} \cap \text{BBC}) = 11$

(b) $n(\text{MTV} \cup \text{BBC}) = 74$

(c) $n(\text{CNN} \cap \text{BBC} \cap \text{MTV}') = 2$

(d) $n(\text{MTV} \cup \text{CNN} \cap \text{BBC}') = 77$

2.2 Probability

Suppose a fair coin is tossed. There are 2 possible outcomes: a head or a tail. The probability of getting a head or a tail is $\frac{1}{2}$.

Sample Space: list of all possible outcomes

Outcome: possible results of an experiment (i.e. a well-defined process from which observations may be made)
Example: outcomes of rolling a fair die would be the 6 possible faces 1, 2, 3, 4, 5, 6

Event: the outcomes that meet the particular requirement (i.e. an event is a subset of the sample space)

Example: Toss a coin four times. A typical outcome is HTTH. The sample space is the set of all 16 strings of four H's and T's. Then, "exactly 2 heads" is an event. Call this event A.

A = {HHTT, HTHT, HTTH, THHT, THTH, TTHH}

The probability of A is defined by $P(A) = \frac{n(A)}{n(U)}$ where n(U) is the number of all possible outcomes, and n(A) is the number of outcomes that belong to the event A.

Using the above example of tossing a coin 4 times: $P(A) = \frac{n(A)}{n(U)} = \frac{6}{16} = \frac{3}{8}$

Example 1:
Experiment: drawing a card at random from a deck of cards
Event: an ace is drawn
Find the probability that an ace is drawn from a deck of cards.

Example 2: Find the probability of rolling either a two or a four with a die.

Example 3: Find the probability that a number picked at random between 1 and 10 inclusive is:

a) even

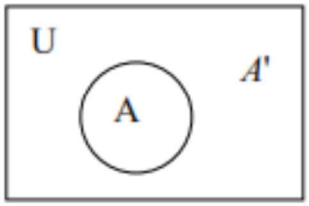
b) a perfect square

Probability Rules:

Rule 1. The probability P(A) of any event A satisfies $0 \leq P(A) \leq 1$. An event with probability 0 never occurs, and an event with probability 1 occurs on every trial.

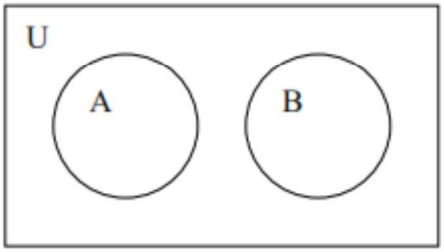
Rule 2. If U is the sample space in a probability model, then P(U) = 1.

Rule 3. The **complement** of any event A is the event that A does not occur, written as A' (or not A). The **complement rule** states that $P(A') = 1 - P(A)$



Rule 4. Two events A and B are **disjoint** if they have no outcomes in common and so can never occur simultaneously. If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$

This is the **addition rule for disjoint events**.



Example 4: What is the probability of not selecting a prime number in a random selection from 1 to 20?

2.2 Practice

1. Assuming that births are equally likely on any day of the week, find the probability that the next person you meet was born on a weekday. $\left[\frac{5}{7}\right]$

2. Two fair dice are thrown. Find the probability that the sum of the scores on the two dice is seven. $\left[\frac{1}{6}\right]$

3. What is the probability of a randomly drawn integer from 1 to 40 is **not** a perfect square? $\left[\frac{17}{20}\right]$
4. What is the probability of tossing exactly 2 heads when one coin is tossed three times? $\left[\frac{3}{8}\right]$
5. What is the probability of randomly drawing either a heart or a face card from a standard deck of cards? $\left[\frac{11}{26}\right]$
6. A certain provincial park has 220 campsites. A total of 80 sites have electricity. Of the 52 sites on the lakeshore, 22 of them have electricity. If a site is selected at random, what is the probability that:

   (a)it will on the lakeshore? $\left[\frac{13}{55}\right]$

   (b) it will have electricity? $\left[\frac{4}{11}\right]$

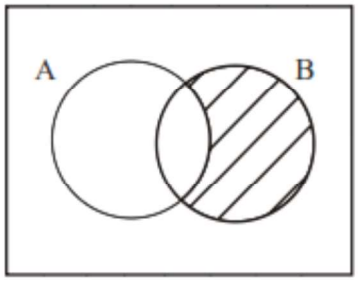
   (c)it will either have electricity or be on the lakeshore? $\left[\frac{1}{2}\right]$

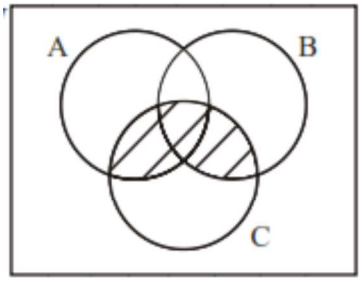
   (d) it will be on the lakeshore and not have electricity? $\left[\frac{3}{22}\right]$

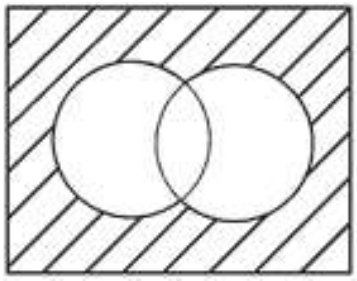
Warm Up

1. Write down an expression to describe the shaded area on the following Venn diagrams:

a) b) c)



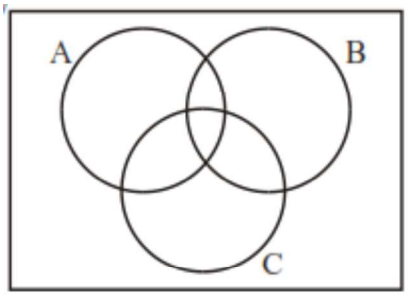


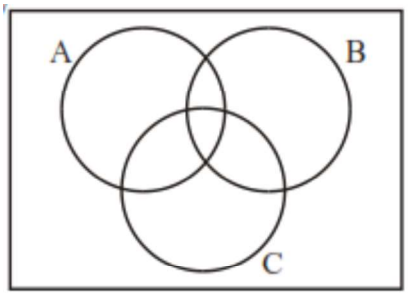


_____ _____ _____

Shade the given region on the corresponding Venn Diagram

a) $(A \cup B \cup C)'$ b) $A \cap C'$





The Vann diagram shows the numbers of pupils in a school according to whether they study the sciences Physics (P), Chemistry (C), Biology (B).

(a) Write down the number of pupils that study chemistry only.

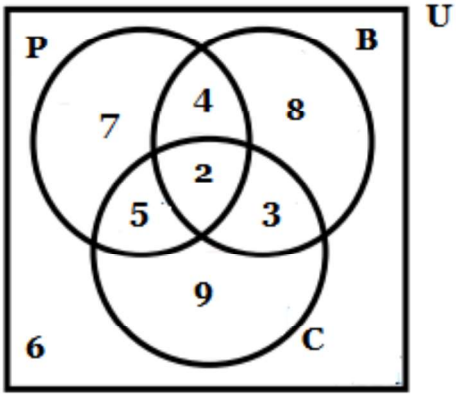
(b) Write down the number of pupils that study exactly two sciences.

(c) Write down the number of pupils that do not study Physics .

(d) Find $n[(P \cup B) \cap C]$.

(e) Find $P(B \cap C)$.

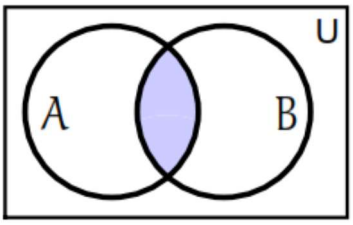
(f) Find $P(P \cup C)$.



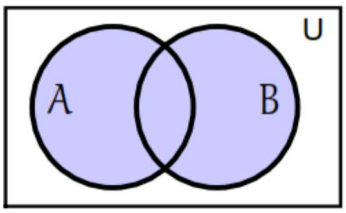
2.3 Combined Events:

Consider two events A and B. Two possible outcomes are:

- $A \cap B$ which means that A and B **both** occur ($\cap$ means the "intersection" of two sets)



- $A \cup B$ which means that A occurs **or** B occurs ($\cup$ means the "union" of two sets); includes the case when A and B both occur.



Principle of Inclusion and Exclusion (2 sets):

For sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

So, $\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$

Other Laws that will help you

$$\boxed{\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ P(A \cup B^c) &= P(A) + P(B^c) - P(A \cap B^c) \end{aligned}}$$

Example 1: At a certain high school, there are twelve players on the school basketball team, and eight players on its volleyball team. How many players could show up at a party if exactly five students play on both teams?

Example 2: In a group of 30 students, 20 use facebook, 12 use twitter, and 5 use neither.

a. Determine the probability of selecting a student at random who uses both facebook and twitter.

b. Determine the probability of selecting a student who uses twitter but does not use facebook.

Example 3: A card is selected at random from an ordinary pack of 52 cards. Find the probability that the card is :

a) a queen b) a diamond c) the queen of diamonds d) either a queen or a diamond.

Example 4: Find the probability of turning up an even number or a number greater than 3 when rolling a die.

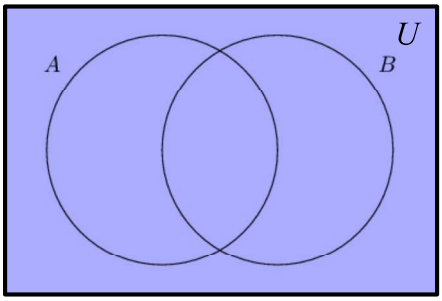
Example 5: As a result of a recent survey conducted by the Substance Abuse League, it was estimated that 85% of a targeted population enjoys an alcoholic beverage at least once a week, 35% of the population smokes at least one cigarette a day, and 25% of the population indulges in both habits. What is the probability that an individual chosen at random from the targeted population either smokes or drinks alcohol?

Example 6: If $P(A) = 0.3$, $P(A \cup B) = 0.4$ and $P(A \cap B) = 0.2$, determine $P(B)$.

Mutually Exclusive Events: Two events A and B are said to be mutually exclusive if they have no outcomes in common (see Rule 4 for disjoint sets) $\boxed{P(A \cup B) = P(A) + P(B) \text{ and } P(A \cap B) = 0}$

Example 1: Given: Experiment: rolling a die
Events: A is that a three turns up; B is that a five turns up
Find the probability that a three or a five turns up when a die is rolled.

Exhaustive Events: Two events are said to be exhaustive if together they include possible outcomes in the sample space. $A \cup B = U$
When A and B are exhaustive, $P(A \cup B) = 1$



Example 2: Given $P(X) = \frac{4}{5}, P(Y) = \frac{1}{2}$ and $P(X \cap Y) = \frac{3}{10}$,show that the events X and Y are exhaustive.

Example 3: If $P(A) = 0.55$, $P(A \cup B) = 0.7$ and $P(A \cap B) = 0.2$, determine $P(B')$.

Example 4: If $P(G') = 5x$, $P(H) = \frac{3}{5}$, $P(G \cup H) = 8x$ and $P(G \cap H) = 3x$, find the value of x.

2.3 Practice

1. A bag contains 7 red discs and 4 blue discs. Ju Shen chooses a disc at random from the bag and removes it. Ramón then chooses a disc from those left in the bag.
   a) Write down the probability that
      (i) Ju Shen chooses a red disc from the bag; [7/11]
      (ii) Ramón chooses a blue disc from the bag, given that Ju Shen has chosen a red disc; [2/5]
      (iii) Ju Shen chooses a red disc and Ramón chooses a blue disc from the bag. [14/55]
   b) Find the probability that Ju Shen and Ramón choose different colored discs from the bag. [28/55]

2. A card is selected at random from a normal playing pack of 52 cards. Find the probability that it is a jack or a spade. [9/13]

3. A survey was carried out in a group of 200 people. They were asked whether they smoke or not. The collected information was organized in the following table.

	Smoker	Non-smoker
Male	60	40
Female	30	70

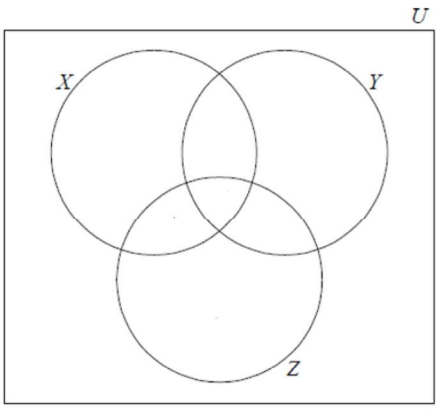
One person from this group is chosen at random.

a) Write down the probability that this person is a smoker. [0.45]

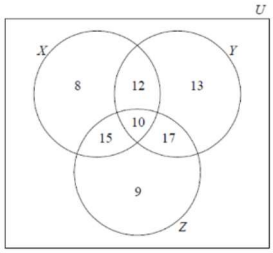
b) Find the probability that this person is a smoker or is male. [13/20]

4. 100 students are asked what they had for breakfast on a particular morning. There were three choices: cereal (X) , bread (Y) and fruit (Z). It is found that 10 students had all three 17 students had bread and fruit only 15 students had cereal and fruit only 12 students had cereal and bread only 13 students had only bread 8 students had only cereal 9 students had only fruit .

   a) Represent this information on a Venn diagram
   b) Find the number of students who had none of the three choices for breakfast. [51%]
   c) Describe in words what the students in the set had for breakfast
   d) Find the probability that a student had at least two of the three choices for breakfast. [0.54]
   e) Two students are chosen at random. Find the probability that both students had all three choices for breakfast. [1/110]



a)



c) (had cereal) and (did not have bread)

Warm-up

In SL Math class, 15 students downhill ski, 20 students snowboard and 5 students do both. There are 37 students. If a student is selected at randomly, state the probability that the student will be:

a) a downhill skier?
b) does not snowboard or do both?
c) a skier given that the student snowboards?

2.4 Conditional Probability and Independent Events

If two events A and B are **mutually exclusive** (both cannot be true at the same time) then
$P(A \cup B) = P(A) + P(B)$
$P(A \cap B) = 0$
However, when A and B are **not mutually exclusive**, $A \cap B = \varnothing$, it can be shown that a more general law applies: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Conditional Probability:
Suppose we have two events A and B, $A|B$ is used to represent that 'A occurs knowing that B has occurred'.

If A and B are events then $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ [P(A|B), read as *the probability of A given B*].

It follows that $P(A \cap B) = P(A|B) \cdot P(B)$ or

$P(A \cap B) = P(B|A) \cdot P(A)$, where events A and B are **not mutually exclusive.**

Example 1: The probability of rain tomorrow is 30% and the probability of snow tomorrow is 40%. The probability of both rain and snow is 10%.

a) Find the probability of rain or snow tomorrow.

b) Find the probability of rain tomorrow if it snows.

Example 2: The probability that John will go to Western is 0.2 and the probability that he will go to another university is 0.5. If John goes to Western, the probability that his girlfriend Jean will follow him and go to Western is 0.75. What is the probability that John and Jean will both go to Western?

Example 3: Two dice are rolled.
a) Determine the probability that doubles occur or a sum of 8 occurs.

b) Determine the probability that the sum of the dice is 4 or at least one of the die is showing a 5.

c) What is the probability of rolling a sum greater than 7 with two dice given that the first die rolled is a 3?

Example 4: Five hundred people rate a new television show. One of these people is selected at random.

	liked it enthusiastically	liked it moderately	hated it	indifferent
male	43	67	91	60
female	57	84	28	70

a) What is the probability that the selected person is not a male?

b) Find the probability that the selected person is both a male and like the show enthusiastically.

c) What is the probability that the selected person was a female who liked the show, either moderately or enthusiastically?

d) If it is known that the selected person liked the show moderately, what is the probability the selected person was a male?

e) If the selected person is a male, what is the probability that he hated the show?

Exit Card!

The following table gives information about Males and Females and whether they prefer Red, Blue, or Yellow.

	Red	Blue	Yellow	Total
Male	20	40	50	110
Female	50	20	20	90
Total	70	60	70	200

Using the table, calculate the following probabilities.

a) P(M) b) P(B ∩M) c) P(B|M) d) P (R ∪ F) e) P (Y^c ∪ F)

2.4 Practice

1. Sue travels the same route to work every day. She has determined that there is a 0.7 probability that she will wait for at least one red light and that there is a 0.4 probability that she will hear her favorite new song on her way to work.

   a) What is the probability that she will not have to wait at a red light and will hear her favorite song? [0.12]

   b) What is the probability that she will have to wait for a red light and hear her favorite song?[0.28]

2. Events A and B have probabilities P(A) = 0.4, P (B) = 0.65, and P(A∪B) = 0.85.
   a) Calculate P(A∩B).
   b) State with a reason whether events A and B are independent.
   c) State with a reason whether events A and B are mutually exclusive.

3. Given that $P(X)=\frac{2}{3}, P(Y|X)=\frac{2}{5}$ and $P(Y|X')=\frac{1}{4}$, find
   a) P(Y ′);
   b) P(X′ ∪ Y ′).

4. The events A and B are such that P(A) = 0.5, P(B) = 0.3, P(A ∪ B) = 0.6.
   a) (i) Find the value of P(A ∩B).
      (ii) Hence show that A and B are not independent.
   b) Find the value of P(B|A).

5. The events B and C are dependent, where C is the event “a student takes Chemistry”, and B is the event “a student takes Biology”. It is known that P(C) = 0.4, P(B|C) = 0.6, P(B|C′) =0.5.
   a) Draw tree diagram.
   b) Calculate the probability that a student takes Biology.
   c) Given that a student takes Biology, what is the probability that the student takes Chemistry?

Warm-Up

1. In a survey, 100 students were asked „do you prefer to watch television or play sport?" Of the 46 boys in the survey, 33 said they would choose sport, while 29 girls made this choice.

	Boys	Girls	Total
Television			
Sport	33	29	
Total	46		100

By completing this table or otherwise, find the probability that

a) a student selected at random prefers to watch television.

b) a student prefers to watch television, given that the student is a boy.

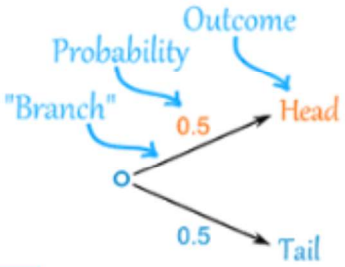
2. The events *B* and *C* are dependent, where *C* is the event "a student takes Chemistry", and *B* is the event "a student takes Biology". It is known that

$$P(C) = 0.4, P(B|C) = 0.6, P(B|C') = 0.5.$$

Create a tree diagram to represent the sample space.

2.5 Probability Tree Diagrams (Independent Events):
Calculating probabilities can be hard, sometimes we add them, sometimes we multiply them, and often it is hard to figure out what to do ... tree diagrams can help

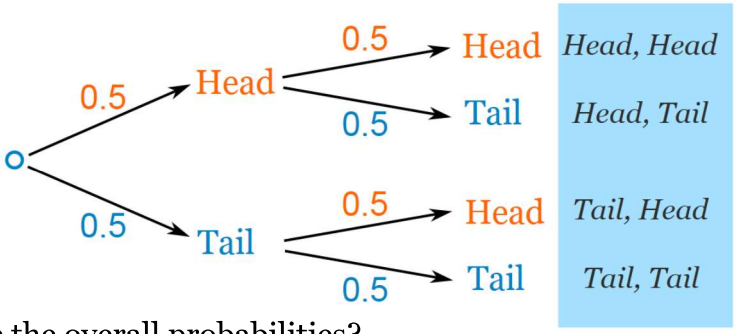
Example: Here is a tree diagram for the toss of a coin:



There are two "branches" (Heads and Tails)

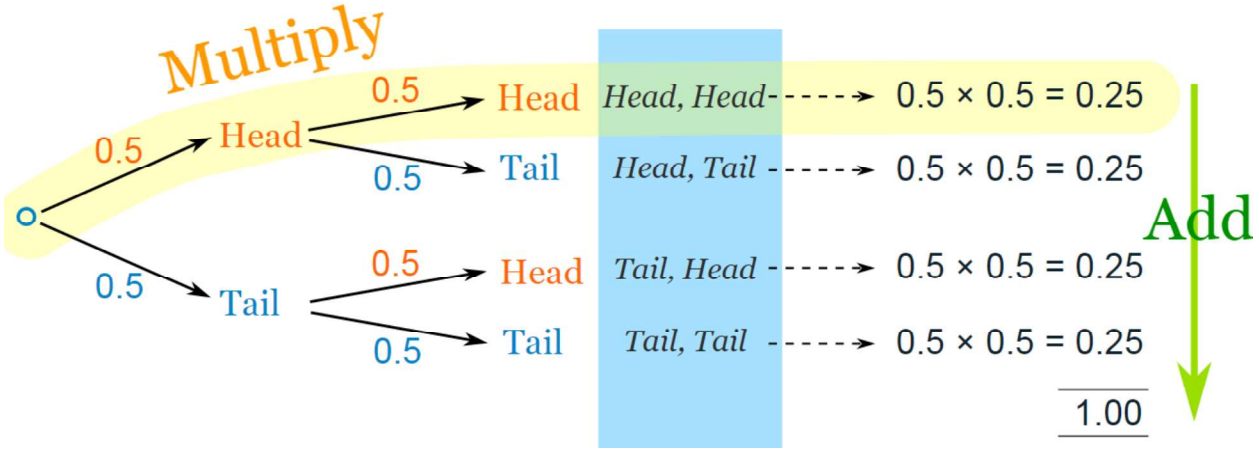
- The probability of each branch is written on the branch
- The outcome is written at the end of the branch

Example: We can extend the tree diagram to two tosses of a coin:



How do we calculate the overall probabilities?

- We **multiply** probabilities **along the branches**
- We **add** probabilities down **columns**



Multiplication Law for Conditional Probability:

I: Suppose we have two **Independent** Events A and B, where the occurrence of either event **does not** affect the probability of the other, then:

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

II: Suppose we have two **Dependent** events A and B, where the occurrence of either event **does** affect the probability of the other, then:

$$P(A \cap B) = P(A) \cdot P(B \mid A)$$

$$P(\text{A then B}) = P(A) \cdot P(\text{B given that A has occurred})$$

Example 4: In a three-child family, assuming equal chance of giving birth to a boy or a girl, what is the probability of
a) having two boys?
b) having at least one girl?

Example 5: In Archery, Charlie hits the target 80% of the time and Jim hits the target 90% of the time. If Charlie shoots and then Jim, what is the probability that the target will get hit exactly once?

Example 6: Cal is not having much luck lately. His car will only start 80% of the time and his motorbike will only start 60% of the time

a) Draw a tree diagram to illustrate this situation
b) Use the tree diagram to determine the chance that
   i) both will start
   ii) Carl can only use his car.

You Try!

1) Two boxes each contain 6 petunia plants that are not yet flowering. Box A contains 2 plants that will have purple flowers and 4 plants that will have white flowers. Box B contains 5 plants that will have purple flowers and 1 plant that will have white flowers. A box is selected by tossing a coin, and one plant is removed at random from it. Determine the probability that it will have purple flowers. [7/12]

2) A box contains 3 red, 2 blue and 1 yellow marble. Find the probability of getting two different color

a) if replacement occurs [11/18]

b) if replacement does not occur.[11/15]

Exit Card!

Use a tree diagram to help you solve the following problem: On any given day, the probability that Rachel will be woken by her alarm clock is $\frac{7}{8}$. If she is woken by her alarm clock, the probability that she will be late for school is $\frac{1}{4}$. If she is not woken by her alarm clock, the probability she will be late for school is $\frac{3}{5}$. Calculate the probability that Rachel will arrive on time for school.

2.5 Practice

1. Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.
   a) Write down the probability that Jack wins on his first throw.
   b) Calculate the probability that Jill wins on her first throw.
   c) Calculate the probability that Jack wins the game.

2. A bag contains 10 red balls, 10 green balls and 6 white balls. Two balls are drawn at random from the bag without replacement. What is the probability that they are of different colours?

3. Bag A contains 2 red and 3 green balls.
   a) Two balls are chosen at random from the bag without replacement. Find the probability that 2 red balls are chosen.

   Bag B contains 4 red and n green balls
   b) Two balls are chosen without replacement from this bag. If the probability that two red balls are $\frac{2}{15}$, show that n=6.

   A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.
   c) Calculate the probability that two red balls are chosen.
   d) Given that two red balls are chosen, find the probability that a 1 or a 6 was obtained on the die.

4. Box A contains 6 red balls and 2 green balls. Box B contains 4 red balls and 3 green balls. A fair cubical die with faces numbered 1, 2, 3, 4, 5, 6 is thrown. If an even number is obtained, a ball is selected from box A; if an odd number is obtained, a ball is selected from box B.
   a) Calculate the probability that the ball selected was red.
   b) Given that the ball selected was red, calculate the probability that it came from box B.

2.6 Probability Distribution

Example 1: Determine the probability distribution for the sum of the roll of two dice.

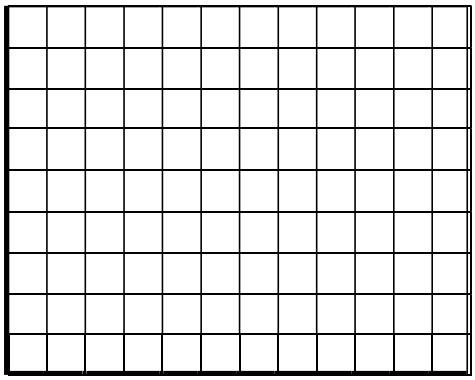
	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Tabulate the probability distribution:

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability											

Note: Sum of probabilities is _____ .

Probability



Sum

To construct a probability model, it is necessary to assign a numerical value to each outcome. This assignment of a numerical value to a real-life occurrence is called the **random variable** and is denoted by X. Example: X was the sum of the two dice.

A random variable, X, is a measurable quantity which can take any value. Its value is the result of a random observation or experiment. Actual measured values are represented by x .

Random variables with outcomes that are assigned integral values (i.e. 0, 1, 2, ...) or certain discrete values are called **discrete random variables**.

Associated with each value of the random variable will be a probability of the value occurring, denoted by $P(X=x)$ which is called the **probability distribution**.

$\sum P(X=x)=1$, Sum of the probabilities of all of the outcomes must equal one.

Example: Determine the probability distribution for the tossing of a coin.
Solution: Let the random variable X be 'the number of tail obtained"

x	0	1
$P(X = x)$		

Note: $P(X = x)$ is the probability that the number of tail is x .

Example 1: For each of these examples, draw a table of possible values of x together with the associated probability $P(X = x)$
a) A box contains 2 red balls and 6 green balls. Two balls are chosen at random with replacement, and X is the number of green balls obtained.
Solution:

Ball 1	Ball 2	# of green balls (x)	Probability

The probability distribution for X is:

x	0	1	2
$P(X = x)$			

b) A fair die has faces labelled 1, 1, 1, 2, 3, 3 and X is the score when the die is thrown.
Solution: The probability distribution for X is:

x			
$P(X = x)$			

c) Two fair dice are thrown and X is the difference between the higher score and the lower score.
Solution:

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

The probability distribution for X is:

x						
$P(X=x)$						

Expectation: The **expected value** is that quantity that you can expect to obtain when the experiment is performed or the mean value of the random variable X and is denoted by $E(X)$. The expected value of X is $E(X)=\sum x\,P(X=x)$

Note: The value of each outcome is multiplied by the probability of its occurrence and then these values are added to obtain the expected value of the experiment.

Example 2: A fair coin is thrown twice and X is the number of tails obtained. Find $E(X)$.

Solution:

1st throw	2nd throw	# of tails (x)	Probability

The probability distribution for X is:

x			
$P(X=x)$			

Example 3: The random variable X can only take the values 1, 2 and 3. Given that the value 3 is twice as likely as each of the values 1 and 2, and values 1 and 2 are equally likely,

a) draw a table of possible values of x together with $P(X=x)$,
b) determine the expectation of X.

Example 4: The random variable X has a probability distribution given by:

$P(X=x)=\frac{x}{k}$, x = 1, 2, 3, 4 (a) Find the value of the constant k. (b) Calculate $E(X)$.

Example: A game is defined by the rules that 2 dice are rolled and the player wins varying amounts depending on the sum of the 2 dice according to the following table:

Sum	2	3	4	5	6	7	8	9	10	11	12
Winnings ($)	10	9	8	7	6	5	6	7	8	9	10

The cost to play the game is $7.50.
a) What can a player expect to win by playing this game?
b) What would be a fair value to pay to play this game?

2.6 Practice

1. If $P(X=0)=\alpha,\ P(X=1)=2\alpha,\ P(X=3)=3\alpha$, find α.

2. The probability distribution of a random variable X is represented by the function

$P(X=x)=\frac{k}{x}$, where $x=1,2,3,4,5,6$. Find a) the value of k b) $P(3\le X\le 5)$

3. A discrete random variable X has a probability distribution defined by the function:

$P(X=x)=\binom{4}{x}\left(\frac{2}{5}\right)^{x}\left(\frac{3}{5}\right)^{4-x}$ where $x=0,1,2,3,4$.

a. Display the function using a table and graph b. Find $P(X=2)$ c. $P(1\le X\le 3)$

4. You draw one card from a standard deck of playing cards. If you pick a heart, you will win $10. If you pick a face card, which is not a heart, you win $8. If you pick any other card, you lose $6. Do you want to play? Explain.

5. The world famous gambler from Philadelphia, Señor Rick, proposes the following game of chance. You roll a fair die. If you roll a 1, then Señor Rick pays you $25. If you roll a 2, Señor Rick pays you $5. If you roll a 3, you win nothing. If you roll a 4 or a 5, you must pay Señor Rick $10, and if you roll a 6, you must pay Señor Rick $15. Is Señor Rick loco for proposing such a game? Explain.

6. You pay $10 to play the following game of chance. There is a bag containing 12 balls, five are red, three are green and the rest are yellow. You are to draw one ball from the bag.You will win $14 if you draw a red ball and you will win $12 is you draw a yellow ball. How much do you expect to win or loss if you play this game 100 times?

7. A detective figures that he has a one in nine chance of recovering stolen property. His out- of-pockets expenses for the investigation are $9,000. If he is paid his fee only if he recovers the stolen property, what should he charge clients in order to breakeven?

8. At Tucson Raceway Park, your horse, Soon-to-be-Glue, has a probability of 1/20 of coming in first place, a probability of 1/10 of coming in second place, and a probability of ¼ of coming in third place. First place pays $4,500 to the winner, second place $3,500 and third place $1,500. Is it worthwhile to enter the race if it costs $1,000?

9. Your company plans to invest in a particular project. There is a 35% chance that you will lose $30,000, a 40% chance that you will break even, and a 25% chance that you will make $55,000. Based solely on this information, what should you do?

10. A manufacturer is considering the manufacture of a new and better mousetrap. She estimates the probability that the new mousetrap is successful is $\frac{3}{4}$. If it is successful it would generate profits of $120,000. The development costs for the mousetrap are $98,000. Should the manufacturer proceed with plans for the new mousetrap? Why or why not?

Warm Up

Allen and Emily, each throw two fair cubical dice simultaneously. The score for each player is the sum of the two numbers shown on their respective dice.

(a) (i) Calculate the probability that Allen obtains a score of 9.

(ii) Calculate the probability that Allen and Emily both obtain a score of 9.

(b) (i) Calculate the probability that Allen and Emily obtain the same score.

(ii) Deduce the probability that Allen's score exceeds Emily's score.

(c) Let X denote the largest number shown on the four dice.

(i) Show that $P(X \leq x) = \left(\frac{x}{6}\right)^4$, for $x = 1,2,3,4,5,6$

(ii) Copy and complete the following probability distribution table.

x	1	2	3	4	5	6
P(X = x)	$\frac{1}{1296}$	$\frac{15}{1296}$				$\frac{671}{1296}$

(iii) Calculate E(X).

2.7 Binomial expansions

Recall: A binomial expression has 2 terms.

Consider the expansion of binomial expressions of the form

$n=0$	$(a+b)^0 = 1$	1 term
$n=1$	$(a+b)^1 = a+b$	2 terms
$n=2$	$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$	3 terms
$n=3$	$(a+b)^3 = (a+b)^2(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$	4 terms
$n=4$	$(a+b)^4 = (a+b)^3(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	5 terms

and so on.

Note:
1) All expansions proceed in decreasing powers of a and increasing powers of b, their exponents always adding to the exponent on the binomial.
2) All the coefficients are precisely the numbers in the Pascal's triangle (see below).
3) The number of terms in the expansion is n+1.
4) In each term, the sum of the exponents is n .

The coefficients of the terms can be written as a triangular array of numbers known as Pascal's triangle.

$n=0$					1					
$n=1$				1		1				
$n=2$			1		2		1			
$n=3$		1		3		3		1		
$n=4$	1		4		6		4		1	
$n=5$										
$n=6$										

A general expression for the coefficient of the $(r+1)^{th}$ term in the expansion of $(a+x)^n$ is:

$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ where $n!=n(n-1)(n-2)\ldots 3\times 2\times 1$ (**called n factorial**)

and where, by definition, $0!=1$. Example: $5! = 5\times 4\times 3\times 2\times 1$

Note: $\binom{n}{r}$ read as "n choose r" appears on calculators as ${}^nC_r$ or nCr .

General formula for the expansion of $(a+x)^n$ where $n \in Z^+$:

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + \binom{n}{n}x^n$$

t_1 t_2 t_3 t_4 t_{r+1} t_{n+1}

Note: There are n+1 terms. The above expansion is often known as the **binomial theorem**.

General formula for the expansion of $(1+x)^n$ where $n \in Z^+$

$$(1+x)^n = \binom{n}{0}1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots\ldots + \binom{n}{n}x^n$$

The general term in the expansion of $(a+x)^n$ is $t_{r+1} = \binom{n}{r}a^{n-r}x^r$.

Example 1: Find the coefficients of the x^3 and x^5 terms in the expansion of $(1+x)^6$.

Example 2: Write down the expansions of:
a) $(a+b)^7$

b) $(x-2)^6$

Example 3: Use the binomial theorem to find the first four terms in the expansion of $\left(2x - \frac{1}{x}\right)^9$.

Example 4: Determine the coefficient of x^4 in the expansion of $(2-3x)^7$.

Example 5: Calculate the value of the constant a if the coefficient of the x^3 term in the expansion $(a+2x)^4$ is 160.

Approximations: Use a binomial expansion to provide a good approximation to decimal quantities raised to a high power.

Example 6: Expand $(1+4x)^{14}$ in ascending powers of x, up to and including the 4^{th} term. Hence, evaluate $(1.0004)^{14}$, correct to 4 decimal places.

Exit Card!

1. The first three terms in the expansion of $(1 - ax)^n$ are $t_1 = 1$, $t_2 = -12x$, and $t_3 = 63x^2$. Use the general term to determine a and n.

2. What is the numerical coefficient for the term containing x^2y^3 in the expansion of $(x - 3y)^5$?

2.7 Practice

1. Expand and simplify the first three terms of $\left(x^2 - 3\sqrt{x}\right)^{10}$. $\left[x^{20} - 30\sqrt{x^{37}} + 405x^{17} + \ldots\right]$

2. In the expansion of $\left(x^2 + \frac{1}{\sqrt{x}}\right)^{10}$ find:

a) the constant term [45]

b) the middle term $\left[252x^{\frac{15}{2}}\right]$

3. In the expansion of $(1+3x)^n$, the coefficient of the term x^2 is 135n, where $n \in Z^+$. Find n. [31]

4. A game consists of cutting a shuffled deck of cards and receiving the dollar value of the face value of the card. Jacks are considered to be worth 11, queens 12, kings 13 and aces 1. What is the expectation of this game?[7]

5. In a manufacturing process, it is estimated that 0.1% of the products are defective. If a client places an order for 25 of the products, what is the probability that at least one of them is defective?

6. A game consists of rolling a pair of dice 10 times. For each sum that equals 6, 7 or 8 on the 2 dice, you win 1 dollar. If it costs 5 dollars to play the game, is this a fair game?

7. Consider the expansion of the expression $(x^3 - 3x)^6$.

a) Write down the number of terms in this expansion. [7]

b) Find the term in x^{12}.[-540]

Warm Up

Consider the expression of $\left(3x^2 - \frac{k}{x}\right)^9$, $k > 0$ .The coefficient of the term in x^6 is 6048. Find the value of k.

2.8 Binomial distribution

A **binomial distribution** is a discrete distribution defined by 2 parameters, '**the number of trials**', **n**, and the '**probability of a success**', **p**.

Bernoulli Trials show up in lots of places. There are 4 essential features:

1. There must be a fixed number of trials, n

2. The trials must be independent of each other

3. Each trial has exactly 2 outcomes called success or failure

4. The probability of success, p, is constant in each trial

Where do we see this occurring?

- tossing a coin
- looking for defective products rolling off an assembly line
- shooting free throws in a basketball game

We will look at two different types of Problems:

1. calculating the probability of first success after n repeated Bernoulli trials
2. calculating the probability of k successes in n repeated Bernoulli trials

Write as X~B(n, p) to indicate that the discrete random variable X is binomially distributed.

Each trial has a probability, p, of **success** or its complementary event – **failure**, whose probability is denoted by q (where q = 1 – p).

The probability distribution of the number of successes in a sequence of repeated trials is given by:

$$P(X = r) = \binom{n}{r} p^r q^{n-r} \qquad \text{or} \qquad P(X = r) = {}^nC_r \, p^r (1-p)^{n-r}$$

Note: In order to have r successes, we would also obtain (n – r) failures in n trials. Therefore, the probability of obtaining a particular sequence of r successes and (n-r) failures is $p^r(1-p)^{n-r}$. In a sequence of n trials, these r successes may occur in $\binom{n}{r}$.

Example 1: Tabulate the probability distribution for the number of heads in the tossing of a coin three times.

Solution: Let the random variable X be the number of heads. Let a success be that a head is tossed. Hence, $n = 3$ with $p = ½$ and $q = 1 - p = ½$. The probability distribution is the binomial distribution.

# of heads (r)	**Probability *P(X = r)***	$\binom{n}{r} p^r q^{n-r}$

Example 2: A fair coin is thrown ten times. Find the probability that 4 heads will occur.

Example 3: The probability that a particular page in a book contains a misprint is 0.2. Find the probability that of 12 pages in the book

a) 7 of them contain a misprint

b) fewer than 2 of them contain a misprint.

What is the Expected value and Variance of a binomial distribution?

The expectation of the number of successes in a binomial distribution of n trials with probability p of success on each trial is np.
For a binomial distribution X ~ B(n, p) :
E(X)= np and
Variance of a binomial distribution =npq
Formula for variance of a binomial distribution is : **Var(X)=npq**

Example 4: An unbiased die is thrown 20 times.

a) Find the expected number of fours thrown.

b) Find the standard deviation of the distribution.

The binompdf command on GDC

Example 5: A survey determines that 62% of the attendees at a conference have attended a similar conference in the last year. Suppose that 9 attendees are randomly selected. Find the probability that **exactly** 4 have attended a similar conference in the last year.

Answer: _____

1. Press [2nd][Vars] for the [DISTR] menu. Scroll down to binompdf(and press [ENTER].
2. The syntax for the binomial probability density function command is binompdf(***n***,***p***,***c***).

TI83	TI84
Type in 9, 0.62, 4) and then press enter. It will always be in this order: binompdf(n, p, c).  	Fill in the needed information, highlight paste, and then press enter. You will then need to press enter again to get the final answer.  

The binomialcdf command on GDC

The "cdf" in this stands for cumulative. This function will take whatever value we type in, and find the cumulative probability for that value and all the values below it. In other words, this function allows us to calculate the probability of "c or fewer" successes, for some number c.

P(X≤c)=binomcdf(*n*,*p*,*c*)

- ***n***: This is the number of trials.
- ***p***: This is the "success" probability.
- ***c***: This finds the probability of c or fewer "successes."

Example 6: A survey determines that 62% of the attendees at a conference have attended a similar conference in the last year. Suppose that 9 attendees are randomly selected. Find the probability that **6 or fewer of these** attendees have attended a similar conference in the last year.

1. Press [2nd][Vars] for the [DISTR] menu. Scroll down to binomcdf(and press [ENTER].
2. The syntax for the binomial cumulative density function command is binomcdf(***n***,***p***,***c***).

TI83	TI84
Type in 9, 0.62, 6) and then press enter. It will always be in this order: binomcdf(n, p, c).	Fill in the needed information, highlight paste, and then press enter. You will then need to press enter again to get the final answer.
	
	

Binomials Rules

(When to use: binompdf or binomcdf)

To find on the calculator: 2nd, vars, and scroll down

n = sample size
p = percent given
r = number of successes (number looking for)

1. Exact number, such as all, half, none, or a specific number: **binompdf(n,p,r)**
2. No more than, at most, does not exceed: **binomcdf(n,p,r)**
3. Less than or fewer then: **binomcdf(n,p,r-1)**
4. At least, or more, no fewer than , not less than: **1 – binomcdf(n,p,r-1)**
5. More than: **1 – binomcdf(n,p,r)**
6. Between two numbers, where a is the small number and b is the larger number:

binomcdf(n,p,b) – binomcdf(n,p,a-1)

p = success q = 1 - p
μ = np
σ = $\sqrt{npq}$

Example 8:

a) A bag contains 2 red balls and 3 green balls. Two balls are drawn at random from the bag without replacement. Find the probability that exactly 2 red balls are drawn.

b) Find the probability that out of 15 attempts of drawing two balls (replaced after each two drawn), more than 12 of them result in exactly 2 red balls being drawn.

c) What is the expected number of attempts in which exactly 2 red balls will be drawn?

2.8 Practice

1. A box contains 35 red discs and 5 black discs. A disc is selected at random and its colour noted. The disc is then replaced in the box.

(a) In eight such selections, what is the probability that a black disc is selected

(i) exactly once?

(ii) at least once?

(b) The process of selecting and replacing is carried out 400 times.

What is the expected number of black discs that would be drawn?

2. A fair coin is tossed eight times. Calculate

(a) the probability of obtaining exactly 4 heads;

(b) the probability of obtaining exactly 3 heads;

(c) the probability of obtaining 3, 4 or 5 heads.

3. A factory makes calculators. Over a long period, 2% of them are found to be faulty. A random sample of 100 calculators is tested.

(a) Write down the expected number of faulty calculators in the sample.

(b) Find the probability that three calculators are faulty.

(c) Find the probability that more than one calculator is faulty.

4. A factory makes switches. The probability that a switch is defective is 0.04.

The factory tests a random sample of 100 switches.

(a) Find the mean number of defective switches in the sample.

(b) Find the probability that there are exactly six defective switches in the sample.

(c) Find the probability that there is at least one defective switch in the sample.

5. Marian shoots ten arrows at a target. Each arrow has probability 0.4 of hitting the target, independently of all other arrows. Let X denote the number of these arrows hitting the target.

(a) Find the mean and standard deviation of X.

(b) Find $P(X \geq 2)$

Warm-up

1. In a manufacturing process, it is estimated that 0.1% of the products are defective. If a client places an order for 25 of the products, what is the probability that at least one of them is defective?

2. A game consists of rolling a pair of dice 10 times. For each sum that equals 6, 7 or 8 on the 2 dice, you win 1 dollar. If it costs 5 dollars to play the game, is this a fair game?

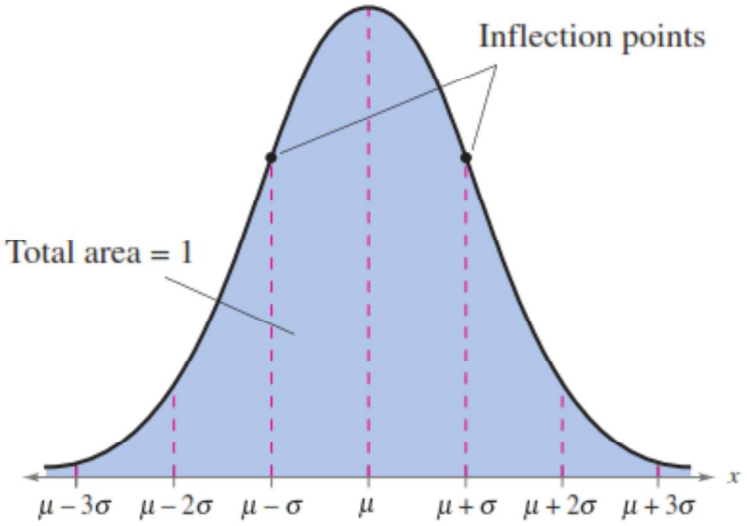
2.9 Normal Distribution

The normal distribution holds an honored role in probability and statistics, mostly because of the central limit theorem, one of the fundamental theorems that forms a bridge between the two subjects. In addition, as we will see, the normal distribution has many nice mathematical properties. The normal distribution is also called the **Gaussian** distribution, in honor of **Carl Friedrich Gauss**, who was among the first to use the distribution.

1777- 1855

A normal distribution is a continuous probability distribution for a random variable x[1]. The graph of a normal distribution is called the normal curve. A normal distribution has the following properties.

1. The mean, median, and mode are equal.
2. The normal curve is bell shaped and is symmetric about the mean.
3. The total area under the normal curve is equal to one.
4. The normal curve approaches, but never touches, the x-axis as it extends farther and farther away from the mean.
5. Between and (in the center of the curve) the graph curves downward. The graph curves upward to the left of and to the right of The points at which the curve changes from curving upward to curving downward are called *inflection points*

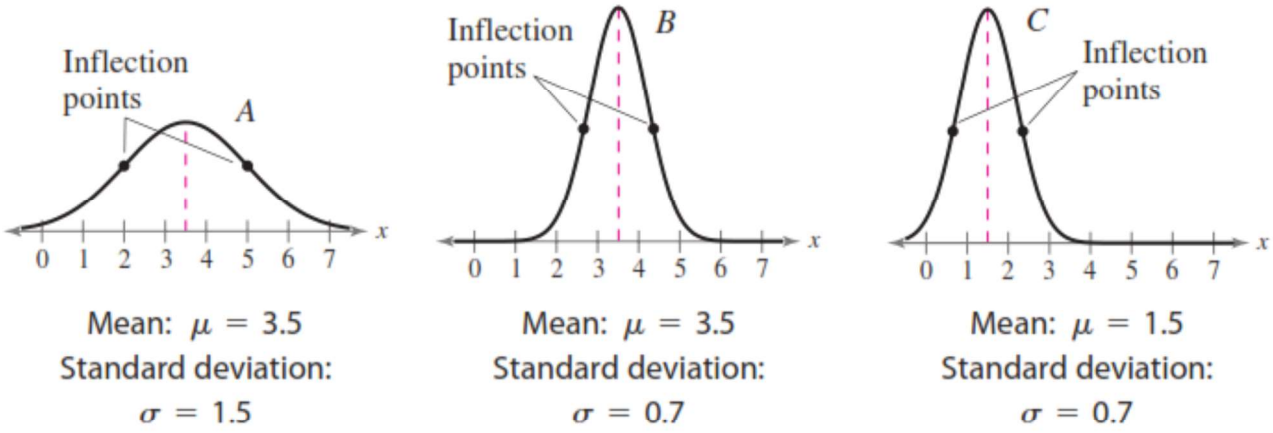


If x is a continuous random variable having a normal distribution with mean μ and standard deviation σ , we write $X \sim N(\mu, \sigma^2)$. You can graph a normal curve using the equation :

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} .$$

[1] See appendix B

A normal distribution can have any mean and any positive standard deviation. These two parameters, and completely determine the shape of the normal curve. The mean gives the location of the line of symmetry, and the standard deviation describes how much the data are spread out.

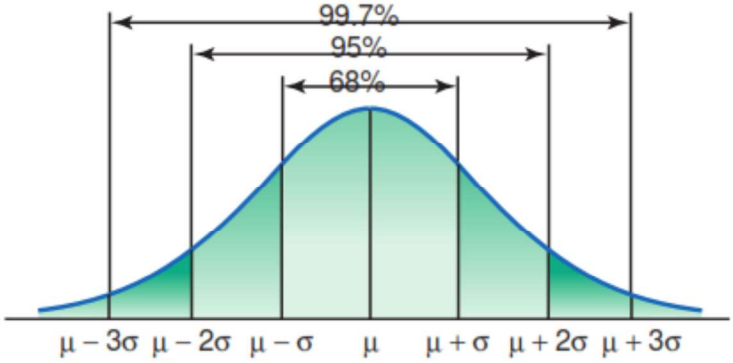


Notice that curve A and curve B above have the same mean, and curve B and curve C have the same standard deviation. The total area under each curve is 1.

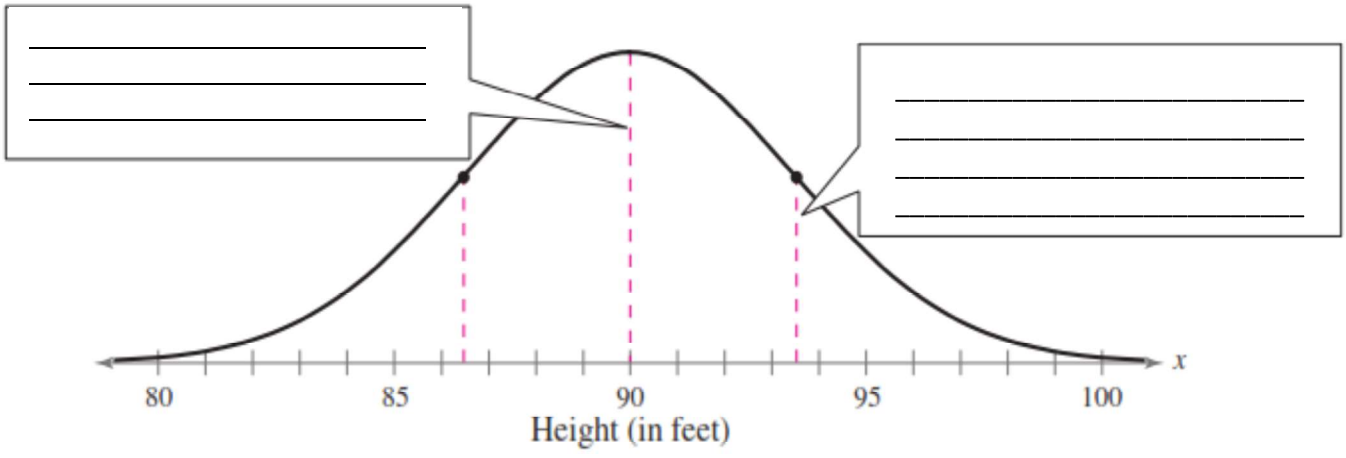
Interpreting Graphs of Normal Distributions

For data that are normally distributed:

- Within 1 standard deviation of the mean, approximately 68% of the data will be found
- Within 2 standard deviations of the mean, approximately 95% of the data will be found
- Within 3 standard deviations of the mean, approximately 99.7% of the data will be found



Example1: The heights (in feet) of fully grown white oak trees are normally distributed. The normal curve shown below represents this distribution. What is the mean height of a fully grown white oak tree? Estimate the standard deviation of this normal distribution.



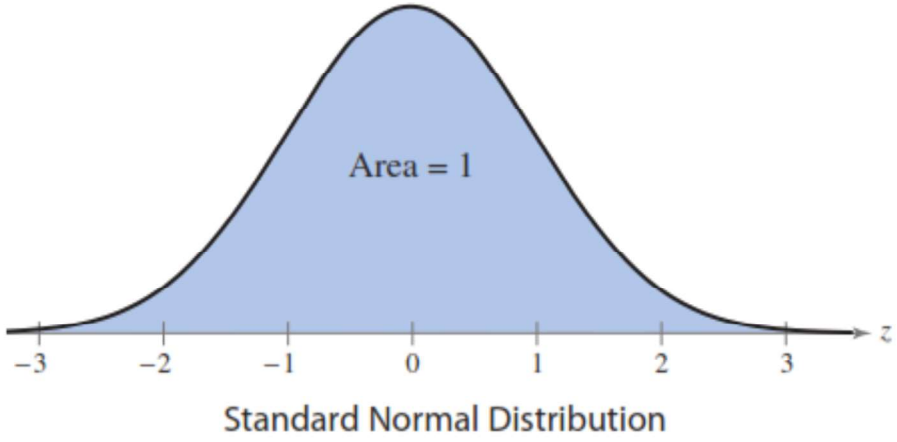
The Standard Normal Distribution

There are infinitely many normal distributions, each with its own mean and standard deviation. The normal distribution with a mean of 0 and a standard deviation of 1 is called the **standard normal distribution.** The horizontal scale of the graph of the standard normal distribution corresponds to scores. A score is a measure of position that indicates the number of standard deviations a value lies from the mean. We can transform an *x*-value to a z-score using the formula $z = \frac{x - \mu}{\sigma}$.

DEFINITION
The standard normal distribution is a normal distribution with a mean of 0 and a standard deviation of 1**.**

Study Tip

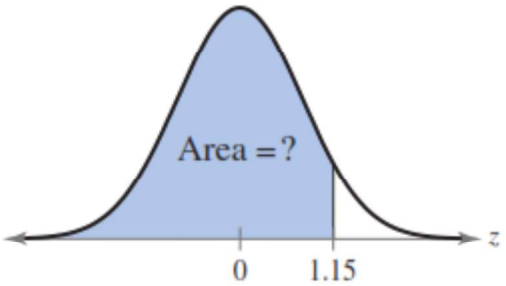
It is important that you know the difference between *x* and *z*. The random variable *x* is sometimes called a raw score and represents values in a *nonstandard* normal distribution, whereas *z* represents values in the *standard* normal distribution.



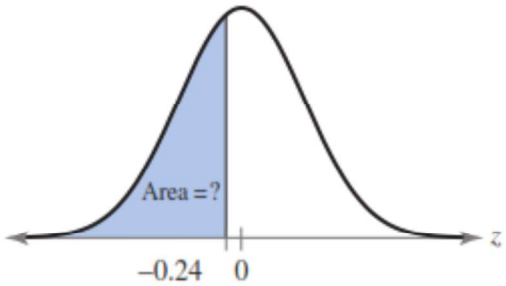
Standard Normal Distribution

After you use the formula given above to transform an x-value to a z-score, you can use the Standard Normal Table in Appendix A. The table lists the cumulative area under the standard normal curve to the left of for scores from -3.49 to 3.49.

Example 2: Using the Standard Normal Table
1. Find the cumulative area that corresponds to a *z*-score of 1.15.
2. Find the cumulative area that corresponds to a *z*-score of -0.24.



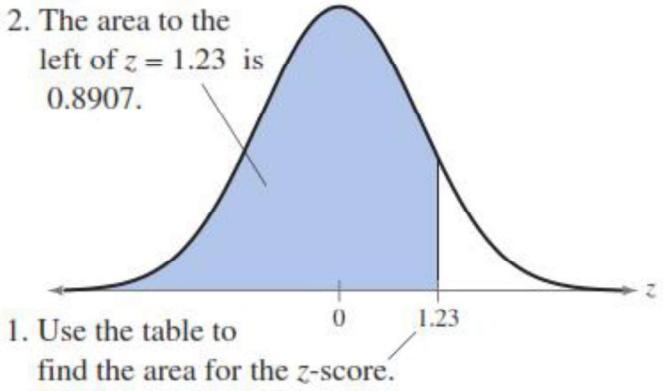
(1)



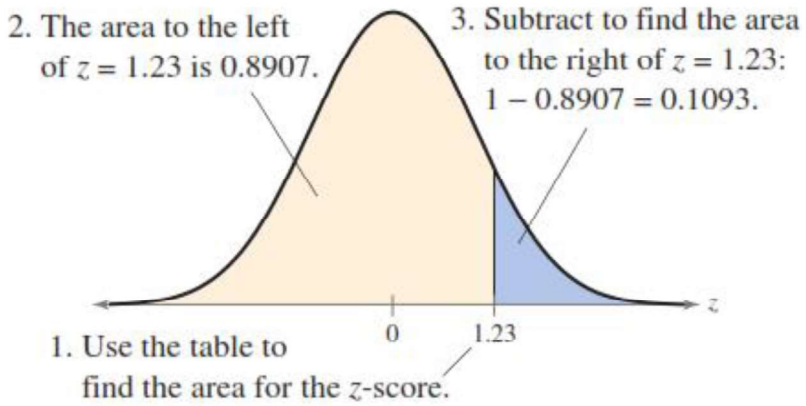
(2)

Probability and Normal Distributions

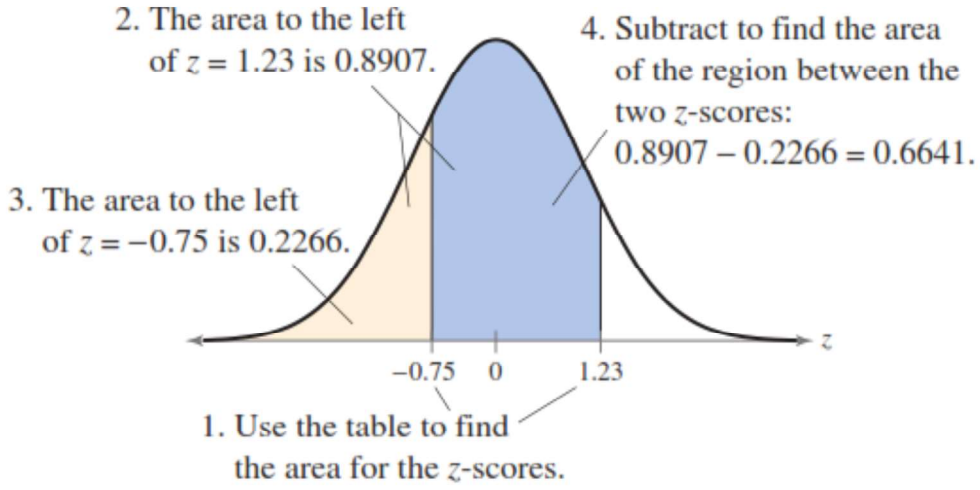
GUIDELINES: Finding Areas Under the Standard Normal Curve
1. Sketch the standard normal curve and shade the appropriate area under the curve.
2. Find the area by following the directions for each case shown.
a. To find the area to the *left* of z find the area that corresponds to z in the Standard Normal Table.



b. To find the area to the *right* of z, use the Standard Normal Table to find the area that corresponds to z. Then subtract the area from 1.

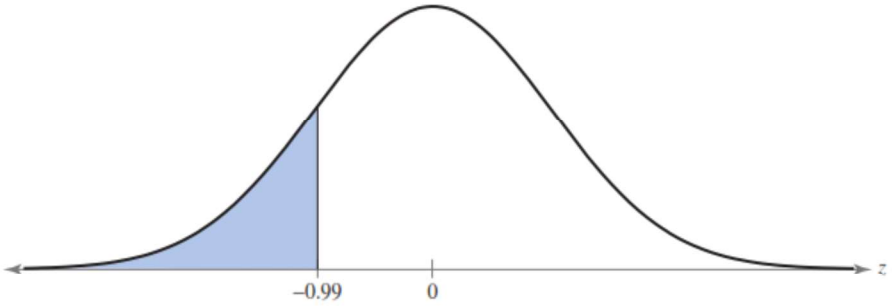


c. To find the area *between* two z-scores, find the area corresponding to each z-score in the Standard Normal Table. Then subtract the smaller area from the larger area.

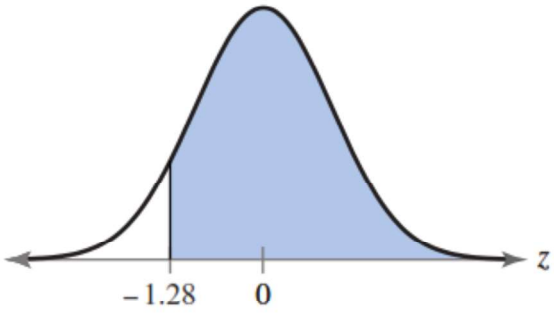


Finding Area Under the Standard Normal Curve

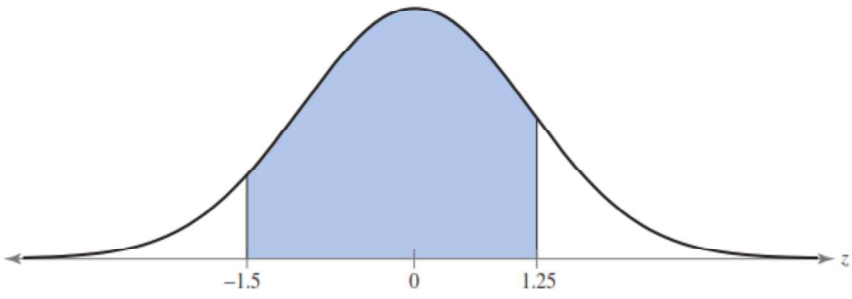
Example3: a) Find the area under the standard normal curve to the left of **z=-0.99**.



b) Find the area under the standard normal curve to the right of **z=-1.28**.



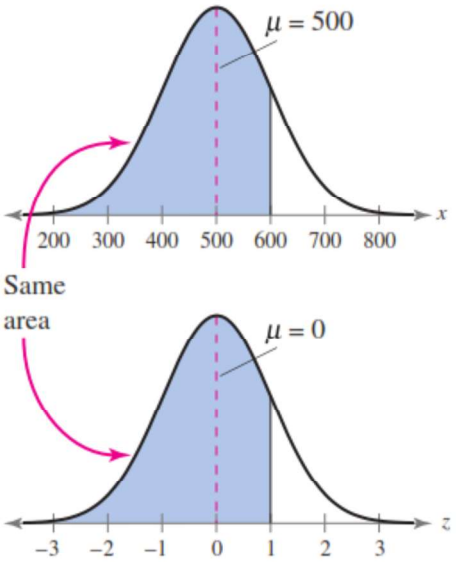
c) Find the area under the standard normal curve between **z =-1.5** and **z= 1.25.**



Theorem: Standardizing a Normal Random Variable

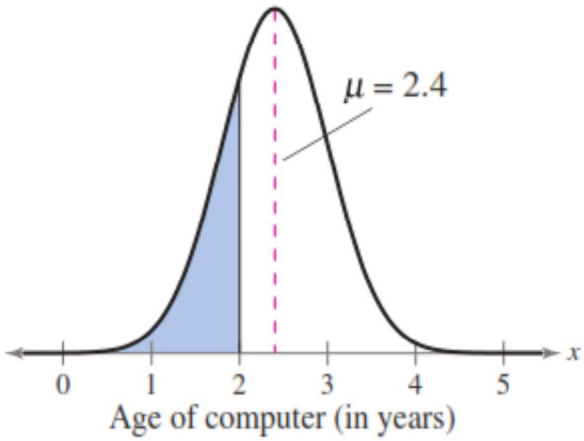
Let $X \sim N(\mu, \sigma^2)$ *and define* $Z = \frac{X - \mu}{\sigma}$. *Then* $Z \sim N(0,1)$.

If a random variable x is normally distributed, you can find the probability that x will fall in a given interval by calculating the area under the normal curve for the given interval. To find the area under any normal curve, first convert the upper and lower bounds of the interval to z-scores. Then use the standard normal distribution to find the area. For instance, consider a normal curve with $\mu=500$ and $\sigma=100$ as shown at the upper left. The value of x one standard deviation above the mean is $\mu + \sigma = 600$ .Now consider the standard normal curve shown at the lower left. The value of z one standard deviation above the mean is $\mu + \sigma = 1$. Because a z-score of 1 corresponds to an x-value of 600, and areas are not changed with a transformation to a standard normal curve, the shaded areas in the graphs are equal.



Finding Probabilities for Normal Distributions

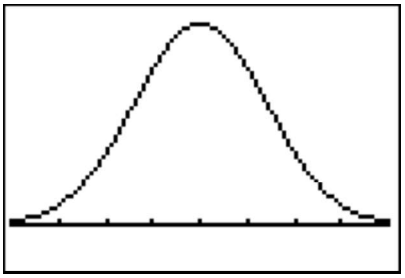
Example 4: A survey indicates that people use their computers an average of 2.4 years before upgrading to a new machine. The standard deviation is 0.5 year. A computer owner is selected at random. Find the probability that he or she will use it for less than 2 years before upgrading.(i.e find $P(X < 2)$ Assume that the variable x is normally distributed.



The normalcdf command on GDC

I. You can easily graph a normal curve with the normal probability distribution function, normalpdf

- To find the normalpdf command, press [2nd][Vars] 1:normalpdf
- Follow these steps to graph a normal curve in Function mode:
  - (i) Make note of the mean, μ , and the standard deviation, σ, of the distribution.
  - (ii) Press Y and define Y1normalpdf(X, μ ,σ). Enter the numerical values of and .
  - (iii) Set an appropriate window.
  - (iv) Press GRAPH . These screens show a normal curve with a mean 3.1 and standard deviation 0.14.



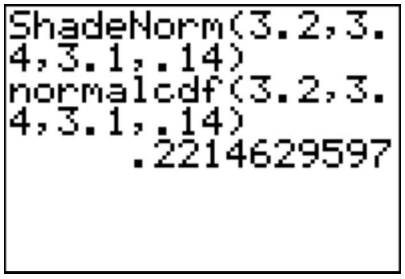
[2.7, 3.5, 0.1, −0.5, 3, 0]

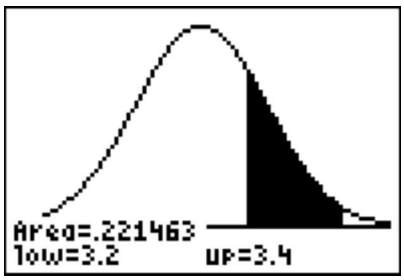
II. Calculating the normal cumulative distribution function, **normalcdf**

- Press [2nd] [VARS] to access the DISTR menu.
- You must choose the **normalcdf** function
- Remember to enter the important numbers into the calculator in order. The rule is: normalcdf(lower,upper,μ ,σ).

III. Graphing Ranges The ShadeNorm command

- press [2nd] [Vars]-> DRAW 1:ShadeNorm
- To use the command, first set an appropriate window. Then, on the Home screen, enter the command in the form ShadeNorm(lower,upper, μ, σ).





- **To find the a- value when given Probability :**
  [2nd] [VARS][Inv Norm] enter value , [Enter]
  - **invnorm (p)**

Example 5: A machine produces an engine part to a mean length of 51. 15 mm and a standard deviation of 0.70 mm. Assuming the distribution of the part lengths to be normal, determine the following:

a) What percentage would be rejected as less than 50 mm long?

b) Verify that 95% of the parts are produced within 2 standard deviations of the mean length.

Example 6: The lifetime of a particular type of battery is normally distributed with a mean life of 40 h and a standard deviation of 0.9 h. Find the probability that a randomly selected battery lasts longer than 42h.

Example 7: A machine produces parts whose lengths are normally distributed with a mean of 20 cm. Given that 8 % of the parts produced by the machine have a length greater than 20.5 cm, find the standard deviation.

Exit Card!

Scores for a civil service exam are normally distributed, with a mean of 75 and a standard deviation of 6.5. To be eligible for civil service employment, you must score in the top 5%. What is the lowest score you can earn and still be eligible for employment

2.9 Practice

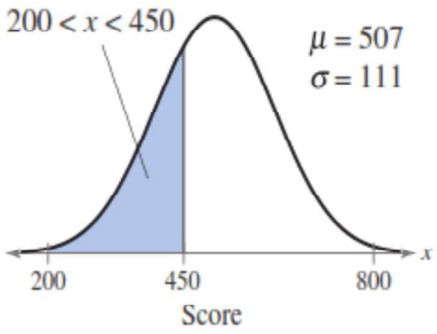
In Exercises 1–6, assume the random variable is normally distributed with mean μ=86 and standard deviation σ=5.Find the indicated probability.

1. $P(x<80)$ **2.** $P(x<100)$ **3.** $P(x>92)$

4. $P(x>75)$ **5.** $P(70<x<80)$ **6.** $P(85<x<95)$

In Exercises 7–12, assume a member is selected at random from the population represented by the graph. Find the probability that the member selected at random is from the shaded area of the graph. Assume the variable *x* is normally distributed.

7.



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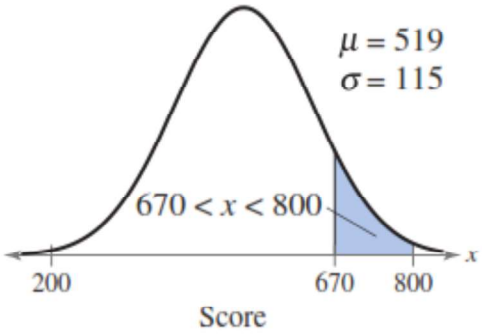

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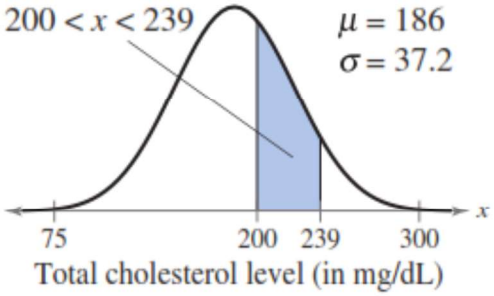

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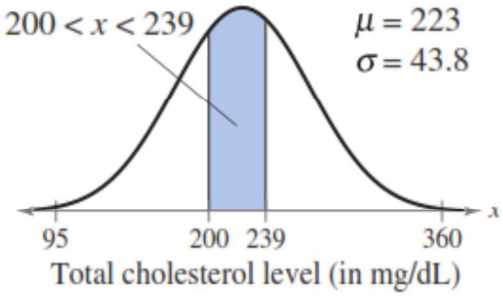

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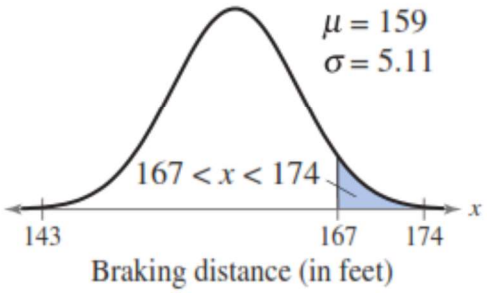

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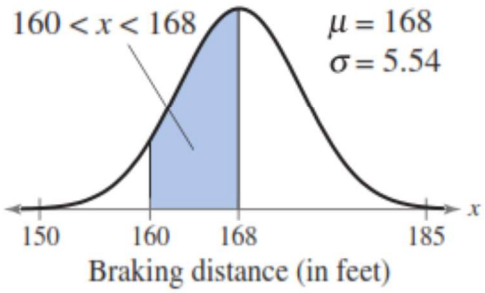
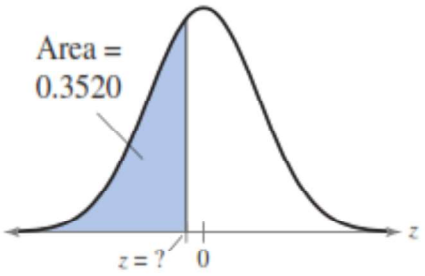


In Exercises 13–18, find the indicated z-score(s) shown in the graph.

13.



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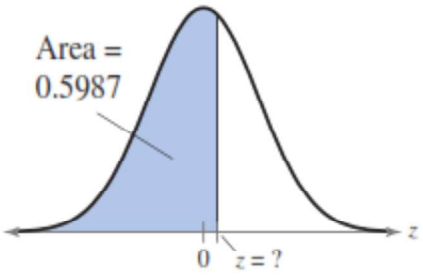

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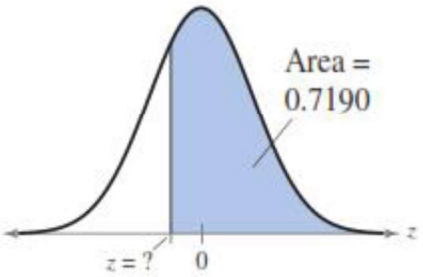

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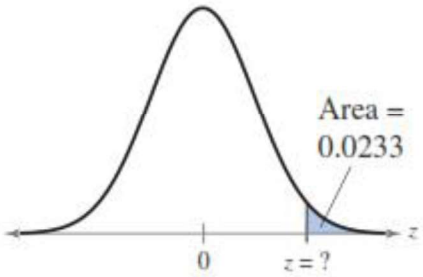

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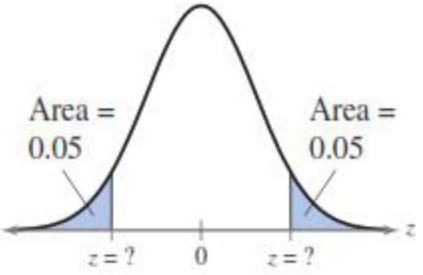

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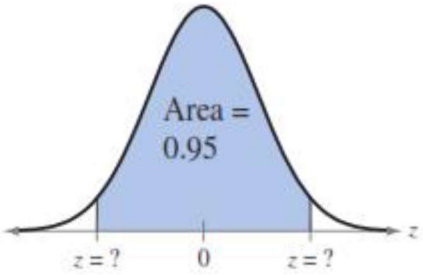


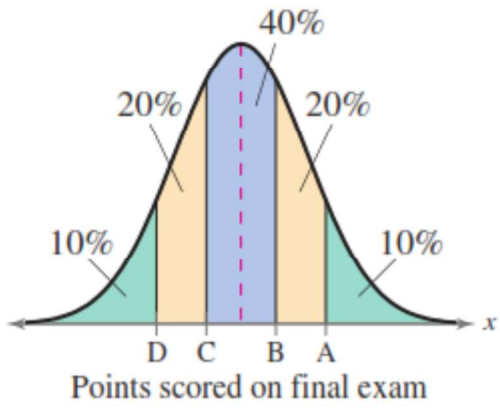
19. The weights of a certain animal are normally distributed with a mean of 36.4 kg and a standard deviation of 4.7 kg. Find the probability that when one of these animals is chosen at random it will have a weight that is:

a) 40.0 kg or less;
b) more than 45.0 kg;
c) between 32.0 kg and 41.0 kg.

20. The random variable X represents the annual consumption, in cubic meters, of water by households in the town of Richmond Hill. X is normally distributed with mean μ and standard deviation σ .Given that 30% of households use more than 200 cubic meters annually and 20% of households use less than 120 cubic meters annually, find the value of μ and the value of σ .

21. In a large section of a statistics class, the points for the final exam are normally distributed with a mean of 72 and a standard deviation of 9. Grades are to be assigned according to the following rule.
    - The top 10% receive As
    - The next 20% receive Bs
    - The middle 40% receive Cs
    - The next 20% receive Ds
    - The bottom 10% receive Fs

Find the lowest score on the final exam that would qualify a student for an A, a B, a C, and a D



Answer

1. 0.1151	2. 0.9974	3. 0.1151	4. 0.9861
5. 0.1144	6. 0.5434	7. 0.3022	8. 0.0878
9. 0.2742	10. 0.3462	11. 0.0566	12. 0.4251
13. -0.38	14. 0.25	15. -0.58	16. 1.99
17. -1.645 , 1.645	18. 1.96		
19. (a) 0.779	(b) 0.0336	(c) 0.0198	

20. $\mu \approx 58.6\ m^3$, $\sigma \approx 169\ m^3$

21. A = 83.52; B = 76.68; C = 67.32; D = 60.48

Warm Up

A company makes containers of yogurt. The volume of yogurt in the containers is normally distributed with a mean of 260ml and standard deviation of 6 ml.

A container which contains less than 250ml of yogurt is underfilled.

(a) A container is chosen at random. Find the probability that it is underfilled.

The company decides that the probability of a container being underfilled should be reduced to 0.02. It decreases the standard deviation to σ and leaves the mean unchanged.

(b) Find σ.

The company changes to the new standard deviation, σ, and leaves the mean unchanged. A container is chosen at random for inspection. It passes inspection if its volume of yogurt is between 250 and 271 ml.

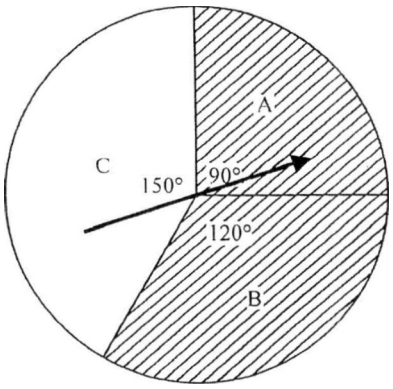
(c) (i) Find the probability that it passes inspection.

(ii) Given that the container is **not** underfilled, find the probability that it passes inspection.

(d) A sample of 50 containers is chosen at random. Find the probability that 48 or more of the containers pass inspection.

Probability Problems

1. The following diagram shows a circle divided into three sectors A, B and C. The angles at the center of the circle are 90°, 120° and 150°. Sectors A and B are shaded as shown. The arrow is spun. It cannot land on the lines between the sectors. Let A, B, C and S be the events defined by



   A: Arrow lands in sector A
   B: Arrow lands in sector B
   C: Arrow lands in sector C
   D: Arrow lands in a shaded region.
   Find:
   a) P(B) b) P(S) c) P(A|S)

2. Let A and B be events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$ and $P(A \cup B) = \frac{7}{8}$, calculate P(A|B).

3. A painter has 12 tins of paint. Seven tins are red and five tins are yellow. Two tins are chosen at random. Calculate the probability that both tins are the same color.

4. Two fair dice are thrown. The sum of two numbers Is S. Find the probability that at least one die shows a 3 given that S is less than 8.

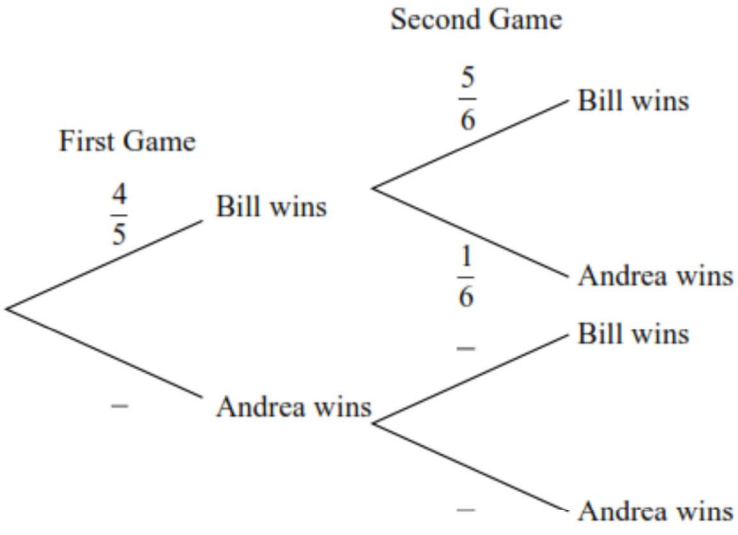
5. A box contains 22 red apples and 3 green apples. Three apples are selected at random, one after the other, without replacement.
   a) The first two apples are green. What is the probability that the third apple is red?
   b) What is the probability that exactly two of the three apples are red?

6. Bill and Andrea play two games of tennis. The probability that Bill wins the first game i $\frac{4}{5}$.

   If Bill wins the first game, the probability that he wins the second game is $\frac{5}{6}$.

   If Bill loses the first game, the probability that he wins the second game is $\frac{2}{3}$.

   a) Complete the following tree diagram



b) Find the probability that Bill wins the first game and Andrea wins the second game.
c) Find the probability that Bill wins at least one game.
d) Given that Bill wins at least one game, find the probability that he wins both games.

7. In a research project on the relation between the gender of 150 science students at college and their degree subject, the following set of data is collected.

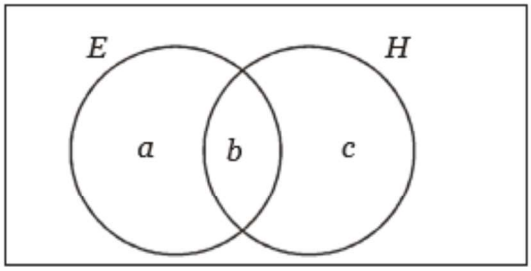
		Degree Subject		
		Biology	Physics	Chemistry
Gender	Male	40	16	35
	Female	15	24	20

a) Find the probability that a student chosen at random is male.
b) Find the probability that a student chosen at random is either male or studies Chemistry.
c) Find the probability that a student chosen at random studies Physics, given that the student is male.

8. Suppose we are given three boxes, Box A contains 10 light bulbs, of which 4 are defective, Box B contains 6 light bulbs, of which 1 is defective and Box C contains 8 light bulbs, of which 3 are defective. We select a box at random and then draw a light bulb from that box at random. What is the probability that the bulb is defective?

9. Coco is a student in an IB World School. The probability that he will be woken by his alarm clock is $\frac{7}{8}$ .If he is woken by his alarm clock the probability he will be late for school is $\frac{1}{4}$ .If he is not woken by his alarm clock the probability he will be late for school is $\frac{3}{5}$. Let W be the event "Coco is woken by his alarm clock". Let L be the event "Coco is late for school".
(a) Calculate the probability that Coco will be late for school.
(b) Given that Coco is late for school what is the probability that he was woken by his alarm clock?

10. In a school of 88 boys, 32 study economics (E), 28 study history (H) and 39 do not study either subject. This information is represented in the following Venn diagram.



(a) Calculate the values a, b, c.
(b) A student is selected at random.
(i) Calculate the probability that he studies **both** economics and history.
(ii) Given that he studies economics, calculate the probability that he does **not** study history.
(c) A group of three students is selected at random from the school.
(i) Calculate the probability that none of these students studies economics.
(ii) Calculate the probability that at least one of these students studies economics.

Answers

1. a) 1/3 b) 7/12 c)3/7

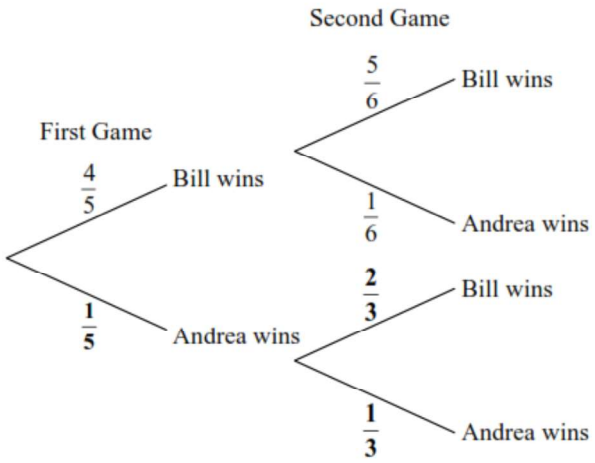
2. ½

3. 31/66

4. 1/3

5. a) 22/23 b) 693/2300

6. a)



b) 2/15 c) 14/15 d) 5/7

7. a) 91/150 b) 111/1550 c) 16/91

8. 0.314

9. a) 47/160 b) 35/47

10. a) a=2 , b=11, c=17

b) i. 1/8 ii. 21/32

c) i. 343/1331 ii. 988/1331

Probability Review Problems

1. For events A and B, the probabilities are P $(A) = \frac{3}{11}$, P $(B) = \frac{4}{11}$. Calculate the value of P $(A \cap B)$ if

(a) P $(A \cup B) = \frac{6}{11}$;

(b) events A and B are independent.

2. Consider events A, B such that P $(A) \neq 0$, P $(A) \neq 1$, P (B) $\neq 0$, and P (B) $\neq 1$.In each of the situations (a), (b), (c) below state whether A and B are mutually exclusive (M); independent (I); neither (N).

(a) P(A|B) = P(A)
(b) P(A $\cap$ B) = 0
(c) P(A $\cap$ B) = P(A)

3. Let A and B be events such that P$(A) = \frac{1}{2}$, P$(B) = \frac{3}{4}$ and P$(A \cup B) = \frac{7}{8}$.

(a) Calculate P$(A \cap B)$.
(b) Calculate P$(A \mid B)$.
(c) Are the events A and B independent? Give a reason for your answer.

4. Let A and B be independent events such that P$(A) = 0.3$ and P$(B) = 0.8$.
(a) Find P$(A \cap B)$.
(b) Find P$(A \cup B)$.
(c) Are A and B mutually exclusive? Justify your answer.

5. Two unbiased 6-sided dice are rolled, a red one and a black one. Let E and F be the events

E : the same number appears on both dice;
F : the sum of the numbers is 10.

Find (a) P(E); (b) P(F); (c) P$(E \cup F)$.

6. Two fair dice are thrown and the number showing on each is noted. The sum of these two numbers is S. Find the probability that
(a) S is less than 8;
(b) at least one die shows a 3;
(c) at least one die shows a 3, given that S is less than 8.

7. A painter has 12 tins of paint. Seven tins are red and five tins are yellow. Two tins are chosen at random. Calculate the probability that both tins are the same color.

8. A class contains 13 girls and 11 boys. The teacher randomly selects four students. Determine the probability that all four students selected are girls.

9. In a survey of 200 people, 90 of whom were female, it was found that 60 people were unemployed, including 20 males.
(a) Using this information, complete the table below.

	Males	Females	Totals
Unemployed			
Employed			
Totals			200

(b) If a person is selected at random from this group of 200, find the probability that this person is
(i) an unemployed female;
(ii) a male, given that the person is employed.

10. A discrete random variable X has a probability distribution as shown in the table below.

x	0	1	2	3
$P(X = x)$	0.1	a	0.3	b

(a) Find the value of $a + b$.
(b) Given that $E(X) = 1.5$, find the value of a and of b.

11. Two ordinary, 6-sided dice are rolled and the total score is noted. Find the probability of getting one or more sixes. Draw a tree diagram to support your answer.

12 . A packet of seeds contains 40% red seeds and 60% yellow seeds. The probability that a red seed grows is 0.9, and that a yellow seed grows is 0.8. A seed is chosen at random from the packet.
(a) Calculate the probability that the chosen seed is red and grows.
(b) Calculate the probability that the chosen seed grows.
(c) Given that the seed grows, calculate the probability that it is red.

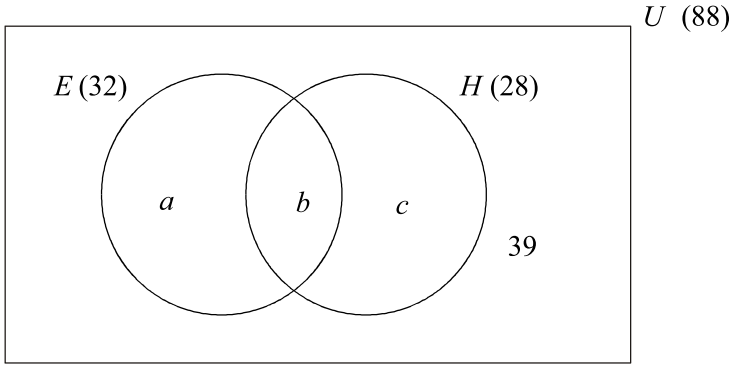
13. A bag contains four apples (A) and six bananas (B). A fruit is taken from the bag and eaten. Then a second fruit is taken and eaten. Find the probability that one of each type of fruit was eaten.

14. In a class, 40 students take chemistry only, 30 take physics only, 20 take both chemistry and physics, and 60 take neither.
(a) Find the probability that a student takes physics given that the student takes chemistry.
(b) Find the probability that a student takes physics given that the student does **not** take chemistry.
(c) State whether the events "taking chemistry" and "taking physics" are mutually exclusive, independent, or neither. Justify your answer.

15. The heights, H, of the people in a certain town are normally distributed with mean 170 cm and standard deviation 20 cm.
(a) A person is selected at random. Find the probability that his height is less than 185 cm.
(b) Given that $P(H > d) = 0.6808$, find the value of d.

16. If $n(U) = 33$, $n(A \cup B) = 29$, $n(A \cap B) = 5$, and $n(B^c) = 23$, Find $n(A^c)$.

17. In a school of 88 boys, 32 study economics (E), 28 study history (H) and 39 do not study either subject. This information is represented in the following Venn diagram.



(a) Calculate the values a, b, c.
(b) A student is selected at random.
(i) Calculate the probability that he studies **both** economics and history.
(ii) Given that he studies economics, calculate the probability that he does **not** study history.
(c) A group of three students is selected at random from the school.
(i) Calculate the probability that none of these students studies economics.
(ii) Calculate the probability that at least one of these students studies economics.

18. A game is played, where a die is tossed and a marble selected from a bag.
Bag M contains 3 red marbles (*R*) and 2 green marbles (*G*).
Bag N contains 2 red marbles and 8 green marbles.
A fair six-sided die is tossed. If a 3 or 5 appears on the die, bag M is selected (*M*).
If any other number appears, bag N is selected (*N*).
A single marble is then drawn at random from the selected bag.

(a) (i) Write down the probability that bag M is selected and a green marble drawn from it.
(ii) Find the probability that a green marble is drawn from either bag.
(iii) Given that the marble is green, calculate the probability that it came from Bag M.

(b) A player wins \$2 for a red marble and \$5 for a green marble. What are his expected winnings?

19. A fair coin is tossed 10 times. Find the probability of getting

i) Exactly 6 heads
ii) at least 2 heads
iii) at most 8 heads

20. The probability distribution of a discrete random variable X is given by $P(X = x) = \frac{x^2}{14}$ $x \in \{1,\ 2,\ k\}$, where $k > 0$. Find the value of k.

21. The heights of trees in a forest are normally distributed with mean height 17 metres. One tree is selected at random. The probability that a selected tree has a height greater than 24 metres is 0.06.

(a) Find the probability that the tree selected has a height less than 24 metres.

(b) The probability that the tree has a height less than D metres is 0.06. Find the value of D

22. A random variable X is distributed normally with a mean of 100 and a variance of 100.

(a) Find the value of X that is 1.12 standard deviations above the mean.

(b) Find the value of X that is 1.12 standard deviations below the mean.

23. Let A and B be independent events, where P(A) = 0.6 and P(B) = x.

(a) Write down an expression for P(A ∩ B).

(b) Given that P(A ∪ B) = 0.8,

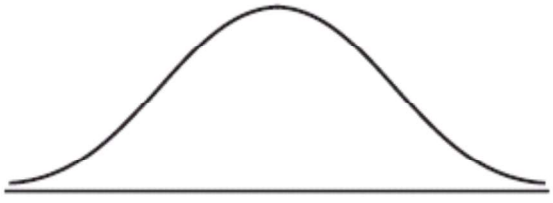
(i) find x;

(ii) find P(A ∩ B).

(c) Hence, explain why A and B are not mutually exclusive

24. Let X be normally distributed with mean 100 cm and standard deviation 5 cm.

(a) On the diagram below, shade the region representing P(X > 105).



(b) Given that P(X < d) = P(X > 105), find the value of d.

(c) Given that P(X > 105) = 0.16 (correct to two significant figures), find P(d < X < 105).

25. In any given season, a soccer team plays 65 % of their games at home. When the team plays at home, they win 83 % of their games. When they play away from home, they win 26 % of their games. The team plays one game.

(a) Find the probability that the team wins the game.

(b) If the team does not win the game, find the probability that the game was played at home.

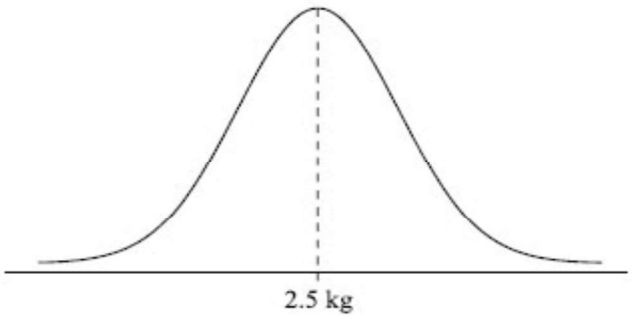
26. The weights of chickens for sale in a shop are normally distributed with mean 2.5 kg and standard deviation 0.3 kg.

(a) A chicken is chosen at random.

(i) Find the probability that it weighs less than 2 kg.

(ii) Find the probability that it weighs more than 2.8 kg.

(iii) Shade the areas that represent the probabilities from parts (i) and (ii).



(iv) **Hence** show that the probability that it weighs between 2 kg and 2.8 kg is 0.7936 (to four significant figures).

(b) A customer buys 10 chickens.

(i) Find the probability that all 10 chickens weigh between 2 kg and 2.8 kg.

(ii) Find the probability that at least 7 of the chickens weigh between 2 kg and 2.8 kg.

27. Find the value of k and n if $(1+kx)^n = 1-4x+7x^2+\ldots$

28. Find the coefficient of x^2 in $(2+x)^4\left(1+\frac{1}{x^2}\right)$.

29. Find the coefficient of x^4 in $\left(3x^2-\frac{2}{x}\right)^5$.

30. Consider the expansion of the expression $(x^3 - 3x)^6$

(a) Write down the number of terms in this expansion.

(b) Find the term in x^{12}