

2.1 Sets & Venn Diagrams

A **set** is a collection of things.

For example, the items you wear is a set: these would include shoes, socks, shirt, pants, etc.



The set of PRIME NUMBERS: $\{2, 3, 5, 7, 11, 13, \dots\}$

The set of WHOLE NUMBERS: $\{0, 1, 2, 3, \dots\}$

Set Theory Symbols and Definitions

Symbol	Name	Definition	Example
$\{ \}$	Set	A collection of elements (objects)	$A = \{2, 7, 8, 9, 15, 23, 35\}$
U	Universal set	set that contains all the elements under discussion for a particular situation	
$A \cap B$	Intersection	Objects that belong to set A and set B	If set $A = \{1, 2, 3\}$ and set $B = \{2, 3, 4\}$, then $A \cap B = \{2, 3\}$
$A \cup B$	Union	Objects that belong to set A or set B	If set $A = \{1, 2, 3\}$ and set $B = \{4, 5, 6\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6\}$
$A \subseteq B$	Subset	Set A is a subset of set B iff every element of set A is in set B	If set $A = \{a, b, c\}$ and set $B = \{a, b, c, d, e\}$, then $A \subseteq B$
A^c or A'	Complement	All objects that do not belong to set A	If the universal set $U = \{a, b, c, d, e\}$ and $A = \{b, e\}$ then $A' = \{a, c, d\}$

Number of elements in set A = $n(A)$.

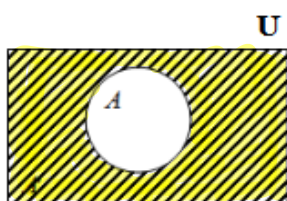
Null set empty set - has no objects = $\emptyset$ or $\{ \}$.

Disjoint sets - have no common elements. (Also referred to as **distinct**.)

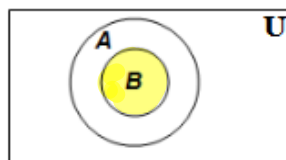
Venn Diagrams are a way of representing sets.

A Venn Diagram usually consists of a rectangle which represents the sample space, and circles within it represent particular events.

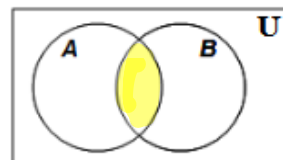
When we have 2 (or more) events we can represent the relationships between the events with the aid of a Venn diagram. The following Venn diagrams show the relationships possible between two events.



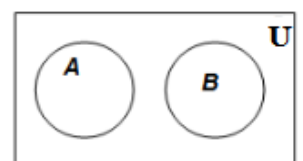
A' is the complement of A



B is a subset of A
All elements of B are in A



A and B have elements in common
 $A \cap B$



A and B have no common elements
A and B are **disjoint sets**
A and B are **mutually exclusive events**
 $A \cap B = \emptyset$

The union of 2 sets ($A \cup B$) contains the elements that are in A or B or both (inclusive or).

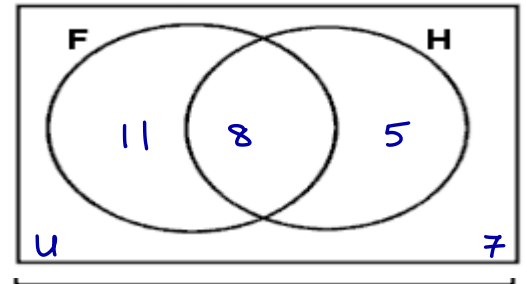
The intersection of 2 sets ($A \cap B$) contains the elements that are in A and B.

There is also an exclusive or where A or B means in A or in B but not in both.

Example This morning I can go to school or I can stay home

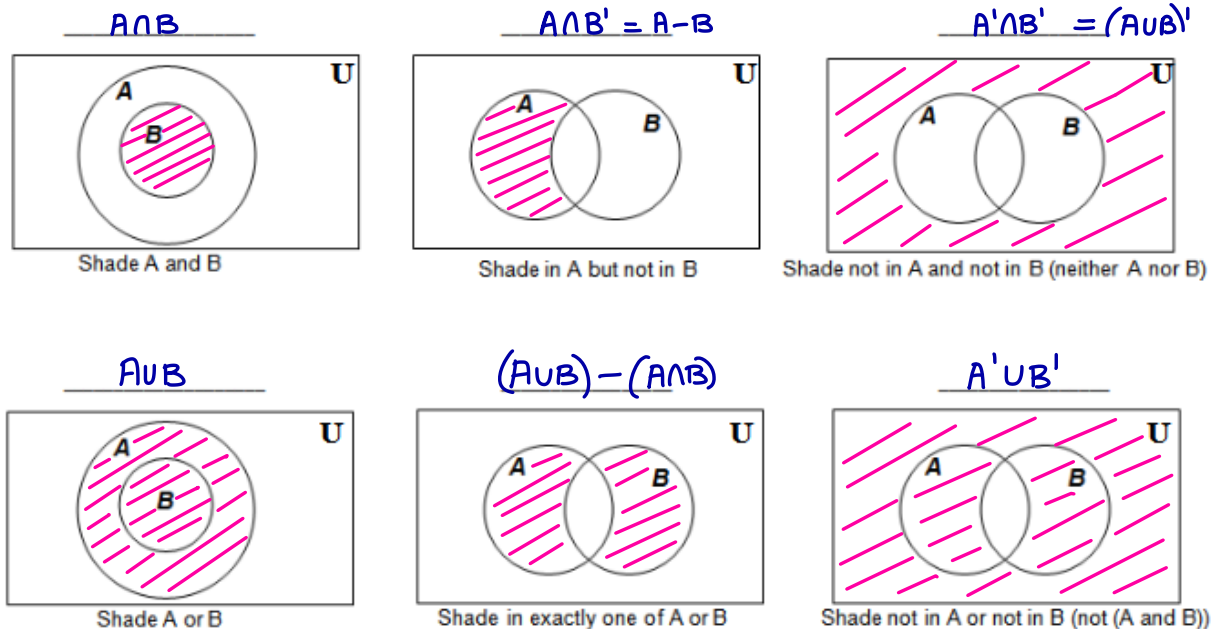
Example 1: In a class there are:

- 8 students who play football and hockey
- 7 students who do not play football or hockey
- 13 students who play hockey
- 19 students who play football



How many students are there in the class? **31 students**

Example 2: Practice shading appropriately and write the proper notation for each scenario.

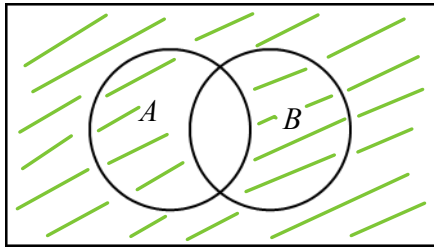


Example 3: Consider the following sample space: $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the event $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4\}$, $C = \{6, 7\}$, $D = \{3, 4, 7, 8\}$. List the elements in the following compound events:

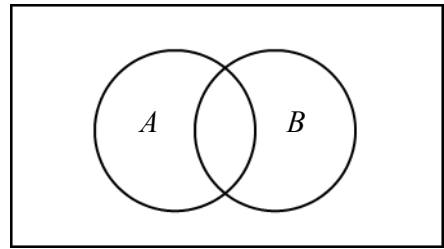
- | | |
|--|--|
| <p>a) $A \cup B$
 $= \{1, 2, 3, 4, 5\}$</p> <p>b) $A \cup D$
 $= \{1, 2, 3, 4, 5, 7, 8\}$</p> <p>c) B'
 $= \{1, 5, 6, 7, 8, 9, 10\}$</p> <p>d) $A \cap B$
 $= \{2, 3, 4\}$</p> <p>e) $A \cap C$
 $= \{ \}$
 null set</p> | <p>f) $A \cap D'$ Note $D' = \{1, 2, 5, 6, 9, 10\}$
 $= \{1, 2, 5\}$</p> <p>g) $A' \cup D'$ Note $A' = \{6, 7, 8, 9, 10\}$
 $= \{1, 2, 5, 6, 7, 8, 9, 10\}$ $D' = \{1, 2, 5, 6, 9, 10\}$</p> <p>h) $(A \cap D)'$
 $= \{1, 2, 5, 6, 7, 8, 9, 10\}$ Note $A \cap D = \{3, 4\}$</p> <p>i) $A' \cap D'$
 $= \{6, 9, 10\}$ Note $A' = \{6, 7, 8, 9, 10\}$
 $D' = \{1, 2, 5, 6, 9, 10\}$</p> <p>j) $(A \cup D)'$
 $= \{6, 9, 10\}$
 Note
 $A \cup D = \{1, 2, 3, 4, 5, 7, 8\}$</p> |
|--|--|

Example 4: De Morgan's Laws: Given two sets, $A, B \subset U$, $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$.
Shade in the Venn Diagrams for:

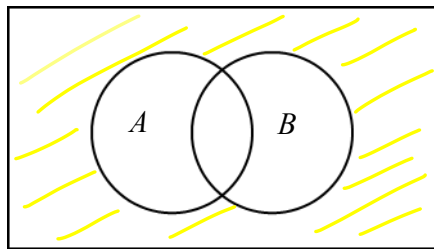
$$A' \cup B'$$



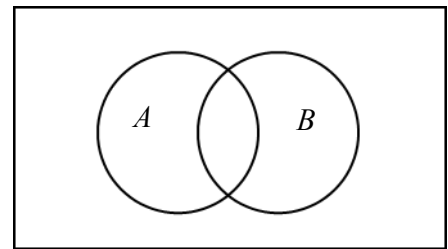
$$(A \cap B)'$$



$$A' \cap B'$$

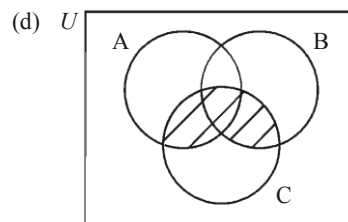
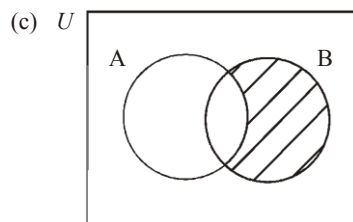
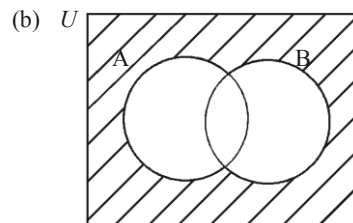
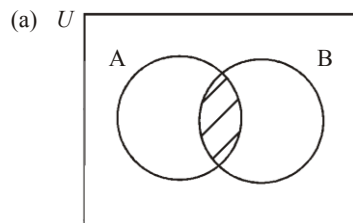


$$(A \cup B)'$$



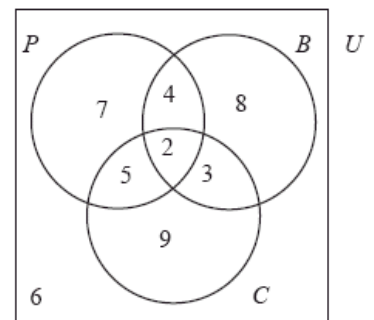
2.1 Practice

- Write down an expression to describe the shaded area on the following Venn diagrams:



- The Venn diagram shows the numbers of pupils in a school according to whether they study the sciences Physics (P), Chemistry (C), Biology (B).

- Write down the number of pupils that study Chemistry only.
- Write down the number of pupils that study **exactly** two sciences.
- Write down the number of pupils that do not study Physics.
- Find $n[(P \cup B) \cap C]$.



3. Let

$M = \{\text{positive integers less than 15}\};$

$X = \{\text{multiple of 2}\};$

$Y = \{\text{multiples of 3}\}.$

- (a) Show, in a Venn diagram, the relationship between the sets M , X and Y .
- (b) List the elements of:
 - (i) $X \cap Y$
 - (ii) $X \cap \neg Y$.
- (c) Find the **number of elements** in the complement of $(X \cup Y)$

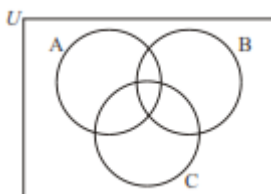
4. The following results were obtained from a survey concerning the reading habits of students.

60% read magazine P
 50% read magazine Q
 50% read magazine R
 30% read magazines P and Q
 20% read magazines Q and R
 30% read magazines P and R
 10% read all three magazines

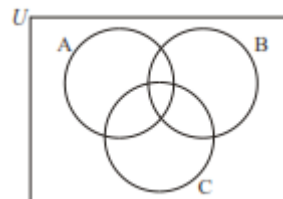
Represent all of this information on a Venn diagram.

5. Shade the given region on the corresponding Venn Diagram.

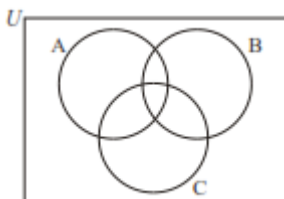
(a) $A \cap B$



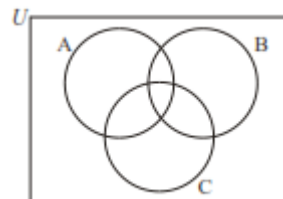
(b) $C \cup B$



(c) $(A \cup B \cup C)'$

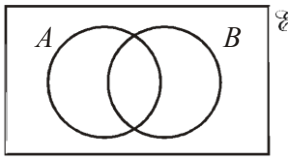


(d) $A \cap C'$

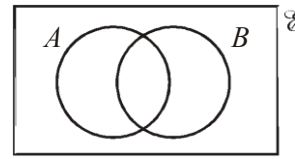


6. In each of the Venn diagrams, shade the region indicated.

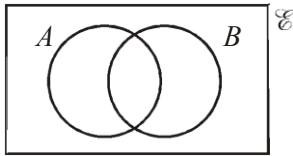
(a) $A \cap B$



(b) The complement of $(A \cap B)$

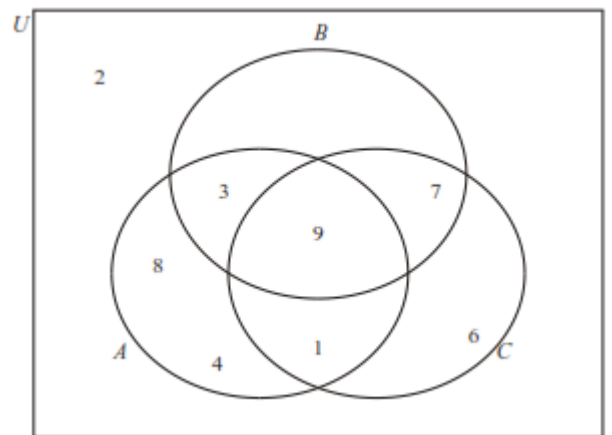


(c) The complement of $(A \cup B)$



7. In the Venn diagram below, A , B and C are subsets of a universal set $U = \{1, 2, 3, 4, 6, 7, 8, 9\}$. List the elements in each of the following sets.

- (a) $A \cup B$
- (b) $A \cap B \cap C$
- (c) $(A' \cap C) \cup B$



8. A group of 30 children are surveyed to find out which of the three sports cricket (C), basketball (B) or volleyball (V) they play. The results are as follows:

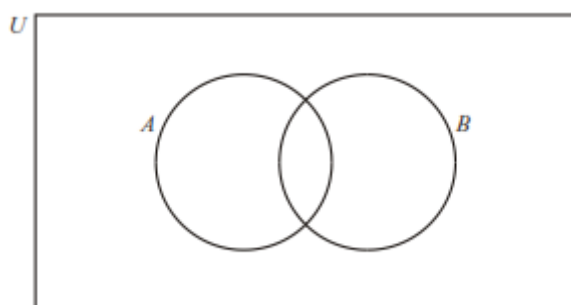
- 3 children do not play any of these sports
- 2 children play all three sports
- 6 play volleyball and basketball
- 3 play cricket and basketball
- 6 play cricket and volleyball
- 16 play basketball
- 12 play volleyball.

- (a) Draw a Venn diagram to illustrate the relationship between the three sports played.
- (b) On your Venn diagram indicate the number of children that belong to each region.
- (c) How many children play only cricket?

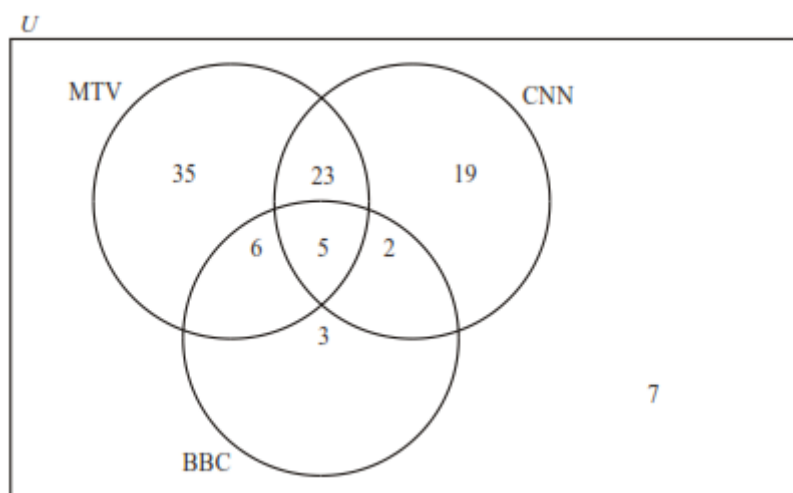
9. The universal set U is defined as the set of positive integers less than 10. The subsets A and B are defined as:

$$A = \{\text{integers that are multiples of 3}\}$$
$$B = \{\text{integers that are factors of 30}\}$$

- (a) List the elements of
- A ;
 - B .
- (b) Place the elements of A and B in the appropriate region in the Venn diagram below.



10. 100 students were asked which television channel (MTV, CNN or BBC) they had watched the previous evening. The results are shown in the Venn diagram below.



From the information in the Venn diagram, write down the number of students who watched

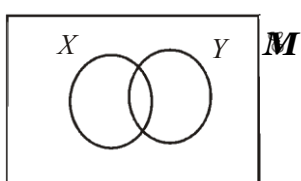
- both MTV and BBC;
- MTV or BBC;
- CNN and BBC but not MTV;
- MTV or CNN but not BBC.

2.1 Practice- Answers

1. (a) $A \cap B$
 (b) $(A \cup B)'$ or $A' \cap B'$
 (c) $A' \cap B$
 (d) $(A \cup B) \cap C$ or $(A \cap C) \cup (B \cap C)$

2. (a) 9
 (b) 12
 (c) $8 + 3 + 9 + 6 = 26$
 (d) $5 + 2 + 3 = 10$

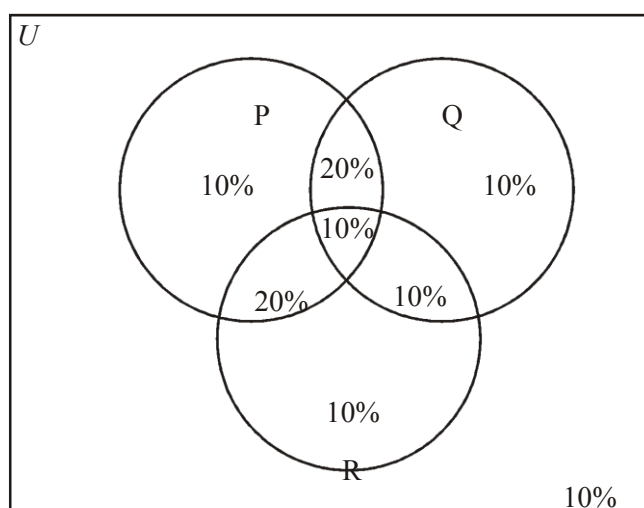
3. (a)



- (b) (i) $(X \cap Y) = \{6, 12\}$
 (ii) $X \cap Y = \{2, 4, 8, 10, 14\}$

- (c) $(X \cup Y)' = (X \cup Y) = \{1, 5, 7, 11, 13\}$
 $n(X \cup Y)' = 5$

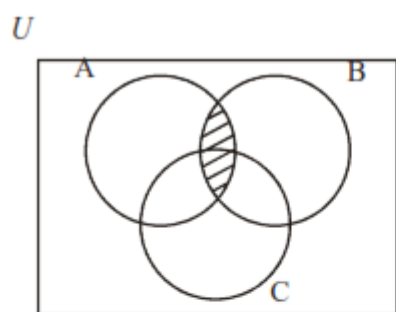
4. (a)



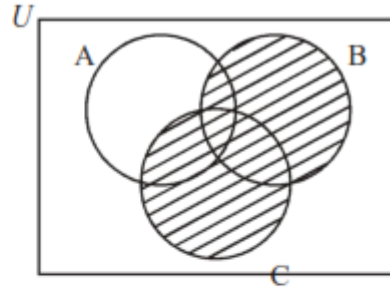
- (b) 50% read exactly two magazines
 (c) 60% read at least two magazines
 (d) 10% do not read any magazines

5.

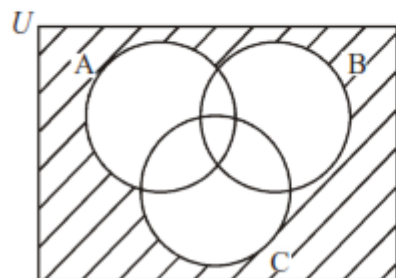
(a)



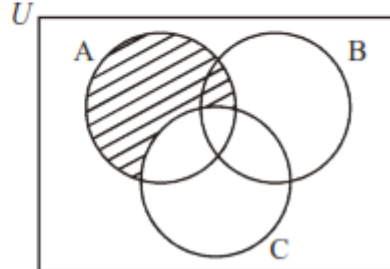
(b)



(c)



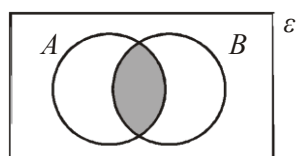
(d)



6.

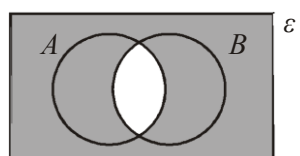
(a)

$$A \cap B$$



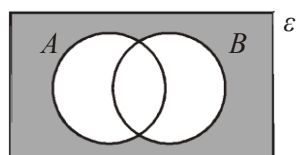
(b)

The complement of $(A \cap B)$



(c)

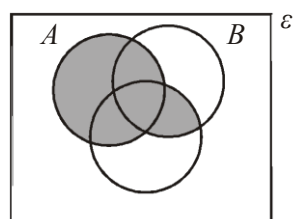
The complement of $(A \cup B)$



C

(d)

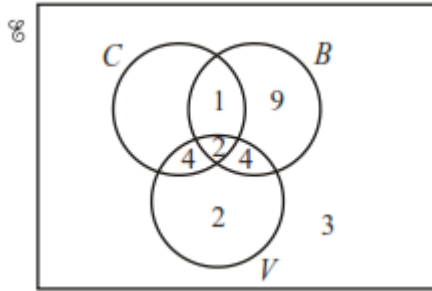
$$A \cup (B \cap C)$$



7.

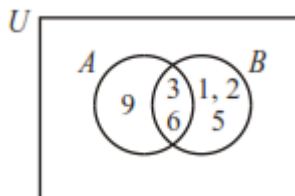
- (a) $A \cup B = \{1, 3, 4, 7, 8, 9\}$
- (b) $A \cap B \cap C = \{9\}$
- (c) $A' = \{1, 3, 4, 7, 8, 9\}$
 $A' \cap C = \{6, 7\}$
 $(A' \cap C) \cup B = \{3, 6, 7, 9\}$

8. (a)
(b)



- (c) $1 + 9 + 4 + 2 + 4 + 2 + 3 = 25$
 $n(C) = 30 - 25$
 $= 5$

9. (a) (i) $A = \{3, 6, 9\}$
(ii) $B = \{1, 2, 3, 5, 6\}$
(b)



10.

- (a) $n(\text{MTV} \cap \text{BBC}) = 11$
- (b) $n(\text{MTV} \cup \text{BBC}) = 74$
- (c) $n(\text{CNN} \cap \text{BBC} \cap \text{MTV}') = 2$
- (d) $n(\text{MTV} \cup \text{CNN} \cap \text{BBC}') = 77$

2.2 Probability

Suppose a fair coin is tossed. There are 2 possible outcomes: a head or a tail. The probability of getting a head or a tail is $\frac{1}{2}$.

Sample Space: list of all possible outcomes

Outcome: possible results of an experiment (i.e. a well-defined process from which observations may be made)

Example: outcomes of rolling a fair die would be the 6 possible faces 1, 2, 3, 4, 5, 6

Event: the outcomes that meet the particular requirement (i.e. an event is a subset of the sample space)

Example: Toss a coin four times. A typical outcome is HTTH. The sample space is the set of all 16 strings of four H's and T's. Then, "exactly 2 heads" is an event. Call this event A.

$$A = \{HHTT, HTHT, HTTH, THHT, THTH, TTHH\}$$

The probability of A is defined by $P(A) = \frac{n(A)}{n(U)}$ where $n(U)$ is the number of all possible outcomes, and $n(A)$ is the number of outcomes that belong to the event A.

Using the above example of tossing a coin 4 times: $P(A) = \frac{n(A)}{n(U)} = \frac{6}{16} = \frac{3}{8}$

Example 1:

Experiment: drawing a card at random from a deck of cards

Event: an ace is drawn

Find the probability that an ace is drawn from a deck of cards.

$$A = \{ \spadesuit A, \heartsuit A, \clubsuit A, \diamondsuit A \}$$

$$\begin{aligned} P(A) &= \frac{n(A)}{n(U)} \\ &= \frac{4}{52} \\ &= \frac{1}{13} \end{aligned}$$

$\therefore$ the probability that an ace is drawn is $\frac{1}{13}$.

Note: Deck of 52 cards $\Rightarrow$
4 diff. suits: , , , 
• Each suit has 13 cards (Ace to King)

Example 2: Find the probability of rolling either a two or a four with a die.

$$U = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{2, 4\}$$

$$\begin{aligned} P(A) &= \frac{n(A)}{n(U)} \\ &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

$\therefore$ the probability of rolling either a two or a four with a die is $\frac{1}{3}$.

Example 3: Find the probability that a number picked at random between 1 and 10 inclusive is:

a) even

$$U = \{1, 2, 3, \dots, 8, 9, 10\}$$

$$A = \{2, 4, 6, 8, 10\}$$

$$P(A) = \frac{n(A)}{n(U)}$$

$$= \frac{5}{10}$$

$$= \frac{1}{2}$$

$\therefore$ probability that an even# is picked is $\frac{1}{2}$.

b) a perfect square

$$B = \{1, 4, 9\}$$

$$P(B) = \frac{n(B)}{n(U)}$$

$$= \frac{3}{10}$$

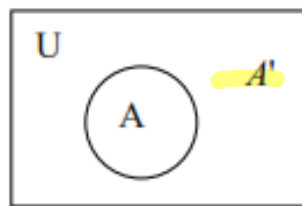
The probability that a perfect square is picked is $\frac{3}{10}$

Probability Rules:

Rule 1. The probability $P(A)$ of any event A satisfies $0 \leq P(A) \leq 1$. An event with probability 0 never occurs, and an event with probability 1 occurs on every trial.

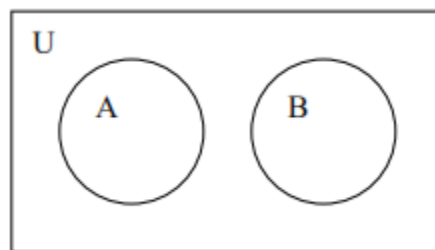
Rule 2. If U is the sample space in a probability model, then $P(U) = 1$.

Rule 3. The **complement** of any event A is the event that A does not occur, written as A' (or **not** A). The **complement rule** states that $P(A') = 1 - P(A)$



Rule 4. Two events A and B are **disjoint** if they have no outcomes in common and so can never occur simultaneously. If A and B are disjoint, then $P(A \cup B) = P(A) + P(B)$

This is the **addition rule for disjoint events**.



Example 4: What is the probability of not selecting a prime number in a random selection from 1 to 20?

$$U = \{1, 2, 3, \dots, 18, 19, 20\}$$

$$A = \{2, 3, 5, 7, 11, 13, 17, 19\}$$

$$\begin{aligned} \therefore P(A) &= \frac{n(A)}{n(U)} \\ &= \frac{8}{20} \\ &= \frac{2}{5} \end{aligned}$$

$$\therefore P(A') = 1 - P(A)$$

$$= 1 - \frac{2}{5}$$

$$= \frac{3}{5}$$

$\therefore$ the probability of not selecting a prime number is $\frac{3}{5}$.

2.2 Practice

1. Assuming that births are equally likely on any day of the week, find the probability that the next person you meet was born on a weekday. $\left[\frac{5}{7} \right]$

$$U = \{S, M, T, W, Th, F, Sa\}$$

$$A = \{M, T, W, Th, F\}$$

$$P(A) = \frac{n(A)}{n(U)}$$

$$= \frac{5}{7}$$

∴ the probability that the next person met was born on a weekday is $\frac{5}{7}$.

2. Two fair dice are thrown. Find the probability that the sum of the scores on the two dice is seven. $\left[\frac{1}{6} \right]$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$P(A) = \frac{n(A)}{n(U)}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

∴ the probability that the sum of the scores on the 2 dice is seven is $\frac{1}{6}$.

3. What is the probability of a randomly drawn integer from 1 to 40 is **not** a perfect square? $\left[\frac{17}{20} \right]$

Let A be the event that the integer is a perfect square

$$A = \{1, 4, 9, 16, 25, 36\}$$

$$P(A') = 1 - P(A)$$

$$= 1 - \frac{n(A)}{n(U)}$$

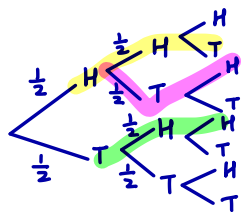
$$= 1 - \frac{6}{40}$$

$$= \frac{34}{40}$$

$$= \frac{17}{20}$$

∴ The probability that the integer is not a perfect square is $\frac{17}{20}$

4. What is the probability of tossing exactly 2 heads when one coin is tossed three times? $\left[\frac{3}{8} \right]$



$$\begin{aligned}
 P(2 \text{ heads}) &= P(HHT) + P(HTH) + P(THH) \\
 &= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) \\
 &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\
 &= \frac{3}{8}
 \end{aligned}$$

The probability of tossing exactly 2 heads is $\frac{3}{8}$

5. What is the probability of randomly drawing either a heart or a face card from a standard deck of cards? $\left[\frac{11}{26} \right]$

Let H be the event that the card is a heart
Let F be the event that the card is a face card

$$\begin{aligned}
 P(H \cup F) &= P(H) + P(F) - P(H \cap F) \\
 &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\
 &= \frac{22}{52} = \frac{11}{26}
 \end{aligned}$$

The probability of drawing either a heart or a face card is $\frac{11}{26}$

6. A certain provincial park has 220 campsites. A total of 80 sites have electricity. Of the 52 sites on the lakeshore, 22 of them have electricity. If a site is selected at random, what is the probability that:

(a) it will be on the lakeshore? $\left[\frac{13}{55} \right]$

(b) it will have electricity? $\left[\frac{4}{11} \right]$

(c) it will either have electricity or be on the lakeshore? $\left[\frac{1}{2} \right]$

(d) it will be on the lakeshore and not have electricity? $\left[\frac{3}{22} \right]$

a) $P(L) = \frac{52}{220}$

$= \frac{13}{55}$

The probability that it will be on the lakeshore is $\frac{13}{55}$

b) $P(E) = \frac{80}{220}$

$= \frac{4}{11}$

The probability that the site will have electricity is $\frac{4}{11}$

c) $P(E \cup L) = P(E) + P(L) - P(E \cap L)$

$= \frac{80}{220} + \frac{52}{220} - \frac{22}{220}$

$= \frac{110}{220}$

$= \frac{1}{2}$

The probability that it will either have electricity or be on the shoreline is $\frac{1}{2}$

d) $P(L \cap E^c) = \frac{52-22}{220}$

$= \frac{30}{220}$

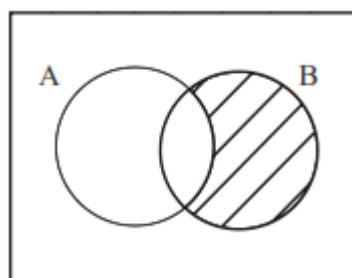
$= \frac{3}{22}$

The probability that it will be on the lakeshore and not have electricity is $\frac{3}{22}$

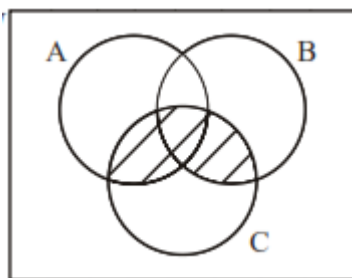
Warm Up

1. Write down an expression to describe the shaded area on the following Venn diagrams:

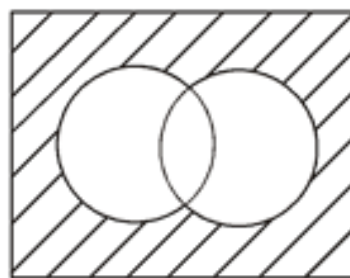
a) b) c)



$$A' \cap B = B - A$$



$$(A \cup B) \cap C'$$

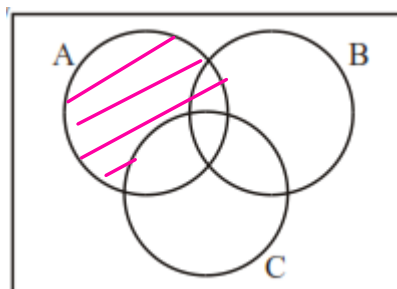
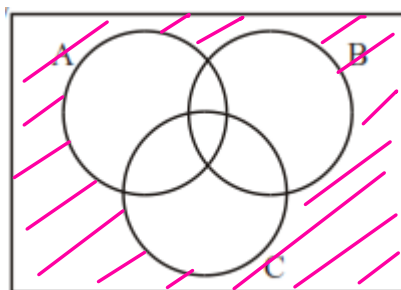


$$(A \cup B)'$$

Shade the given region on the corresponding Venn Diagram

a) $(A \cup B \cup C)'$

b) $A \cap C'$



The Venn diagram shows the numbers of pupils in a school according to whether they study the sciences Physics (P), Chemistry (C), Biology (B).

(a) Write down the number of pupils that study chemistry only. **9 pupils**

(b) Write down the number of pupils that study exactly two sciences. **$5 + 4 + 3 = 12$ pupils**

(c) Write down the number of pupils that do not study Physics .

$$8 + 3 + 9 + 6 = 26 \text{ pupils}$$

(d) Find $n[(P \cup B) \cap C]$.

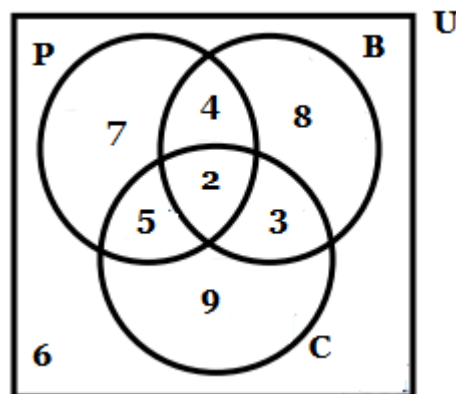
$$5 + 2 + 3 = 10$$

(e) Find $P(B \cap C)$. $P(B \cap C) = \frac{2+3}{44}$

$$= \frac{5}{44}$$

(f) Find $P(P \cup C)$.

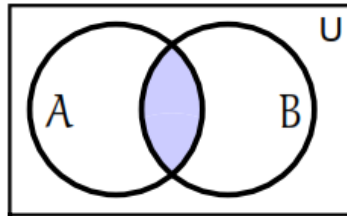
$$P(P \cup C) = \frac{7 + 4 + 5 + 2 + 3 + 9}{44} = \frac{30}{44}$$



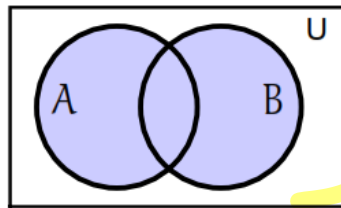
2.3 Combined Events:

Consider two events A and B. Two possible outcomes are:

- $A \cap B$ which means that A and B **both** occur ($\cap$ means the “intersection” of two sets)



- $A \cup B$ which means that A occurs **or** B occurs ($\cup$ means the “union” of two sets); includes the case when A and B both occur.



Principle of Inclusion and Exclusion (2 sets):

For sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

So, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Other Laws that will help you

$$P(A \cap B^c) = P(A) - P(A \cap B)$$

$$P(A \cup B^c) = P(A) + P(B^c) - P(A \cap B^c)$$

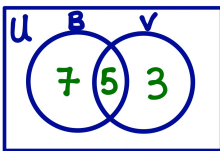
Example 1: At a certain high school, there are twelve players on the school basketball team, and eight players on its volleyball team. How many players could show up at a party if exactly five students play on both teams?

$$n(B \cup V) = n(B) + n(V) - n(B \cap V)$$

$$= 12 + 8 - 5$$

$$= 15$$

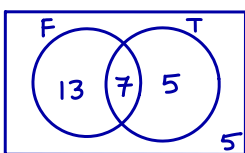
common elements must be subtracted so that they are not counted twice



∴ 15 players could show up at a party.

Example 2: In a group of 30 students, 20 use facebook, 12 use twitter, and 5 use neither.

- Determine the probability of selecting a student at random who uses both facebook and twitter.
- Determine the probability of selecting a student who uses twitter but does not use facebook.



a) $P(F \cup T) = P(F) + P(T) - P(F \cap T)$

$$\frac{25}{30} = \frac{20}{30} + \frac{12}{30} - P(F \cap T)$$

$$P(F \cap T) = \frac{32 - 25}{30} \rightarrow P(F \cap T) = \frac{7}{30}$$

b) The probability of selecting a student who uses twitter and not facebook is $\frac{5}{30}$ or $\frac{1}{6}$

The probability of selecting a student who uses both facebook and twitter is $\frac{7}{30}$

Example 3: A card is selected at random from an ordinary pack of 52 cards. Find the probability that the card is :

- a) a queen b) a diamond c) the queen of diamonds d) either a queen or a diamond.

Let Q denote the event that the card is a queen.

Let D be the event that the card is a diamond.

$$\begin{aligned} \text{a) } P(Q) &= \frac{n(Q)}{n(U)} = \frac{4}{52} = \frac{1}{13} \\ \text{b) } P(D) &= \frac{n(D)}{n(U)} = \frac{13}{52} = \frac{1}{4} \\ \text{c) } P(Q \cap D) &= \frac{1}{52} \end{aligned}$$

$\therefore$ the probability that the card is the queen of diamonds is $\frac{1}{52}$.

$$\begin{aligned} \text{d) } P(Q \cup D) &= P(Q) + P(D) - P(Q \cap D) \\ &= \frac{1}{13} + \frac{1}{4} - \frac{1}{52} \\ &= \frac{4}{13} \end{aligned}$$

$\therefore$ the probability of choosing a queen or a diamond is $\frac{4}{13}$.

Example 4: Find the probability of turning up an even number or a number greater than 3 when rolling a die.

Let A be the event of turning up an even number

Let B be the event of turning up a number greater than 3

$$A = \{2, 4, 6\}$$

$$B = \{4, 5, 6\}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{6} + \frac{3}{6} - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$$

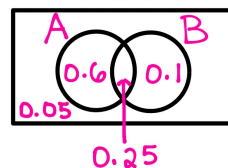
$\therefore$ the probability of rolling an even number or a number greater than 3 is $\frac{2}{3}$.

Example 5: As a result of a recent survey conducted by the Substance Abuse League, it was estimated that 85% of a targeted population enjoys an alcoholic beverage at least once a week, 35% of the population smokes at least one cigarette a day, and 25% of the population indulges in both habits. What is the probability that an individual chosen at random from the targeted population either smokes or drinks alcohol?

Let A be the event that an individual drinks and B , the event that an individual smokes.

$$P(A) = 0.85; P(B) = 0.35; P(A \cap B) = 0.25$$

$$\begin{aligned} \text{Hence, } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.85 + 0.35 - 0.25 \\ &= 0.95 \end{aligned}$$



$\therefore$ the probability that an individual chosen at random either smokes or drinks is 0.95.

Example 6: If $P(A) = 0.3$, $P(A \cup B) = 0.4$ and $P(A \cap B) = 0.2$, determine $P(B)$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.4 = 0.3 + P(B) - 0.2$$

$$P(B) = 0.4 - 0.1$$

$$P(B) = 0.3$$

Mutually Exclusive Events: Two events A and B are said to be mutually exclusive if they have no outcomes in common (see Rule 4 for disjoint sets) $P(A \cup B) = P(A) + P(B)$ and $P(A \cap B) = 0$

Example 1: Given: Experiment: rolling a die

Events: A is that a three turns up; B is that a five turns up

Find the probability that a three or a five turns up when a die is rolled.

Since A and B are mutually exclusive

$$P(A \cup B) = P(A) + P(B)$$

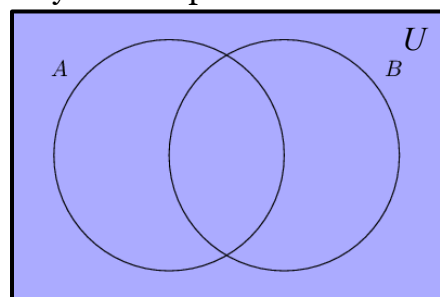
$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{6} = \frac{1}{3}$$

$\therefore$ probability of rolling a 3 or a 5 with a die is $\frac{1}{3}$.

Exhaustive Events: Two events are said to be exhaustive if together they include possible outcomes in the sample space. $A \cup B = U$

When A and B are exhaustive, $P(A \cup B) = 1$



Example 2: Given $P(X) = \frac{4}{5}$, $P(Y) = \frac{1}{2}$ and $P(X \cap Y) = \frac{3}{10}$, show

that the events X and Y are exhaustive.

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

$$= \frac{4}{5} + \frac{1}{2} - \frac{3}{10}$$

$$= \frac{8+5-3}{10}$$

$$= 1$$

$\therefore$ X and Y are exhaustive.

Example 3: If $P(A) = 0.55$, $P(A \cup B) = 0.7$ and $P(A \cap B) = 0.2$, determine $P(B')$.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.7 = 0.55 + P(B) - 0.2$$

$$P(B) = 0.7 + 0.2 - 0.55 = 0.35$$

$$\begin{aligned} \therefore P(B') &= 1 - P(B) \\ &= 1 - 0.35 \\ &= 0.65 \end{aligned}$$

Example 4: If $P(G') = 5x$, $P(H) = \frac{3}{5}$, $P(G \cup H) = 8x$ and $P(G \cap H) = 3x$, find the value of x.

$$P(G) = 1 - P(G')$$

$$= 1 - 5x$$

$$P(G \cup H) = P(G) + P(H) - P(G \cap H)$$

$$8x = (1 - 5x) + \frac{3}{5} - 3x$$

$$8x + 5x + 3x = 1 + \frac{3}{5}$$

$$16x = \frac{8}{5}$$

$$x = \frac{1}{10}$$

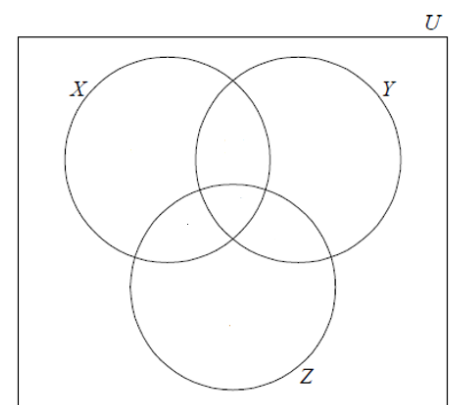
2.3 Practice

- A bag contains 7 red discs and 4 blue discs. Ju Shen chooses a disc at random from the bag and removes it. Ramón then chooses a disc from those left in the bag.
 - Write down the probability that
 - Ju Shen chooses a red disc from the bag; $[7/11]$
 - Ramón chooses a blue disc from the bag, given that Ju Shen has chosen a red disc; $[2/5]$
 - Ju Shen chooses a red disc and Ramón chooses a blue disc from the bag. $[14/55]$
 - Find the probability that Ju Shen and Ramón choose different colored discs from the bag. $[28/55]$
- A card is selected at random from a normal playing pack of 52 cards. Find the probability that it is a jack or a spade. $[9/13]$
- A survey was carried out in a group of 200 people. They were asked whether they smoke or not. The collected information was organized in the following table.

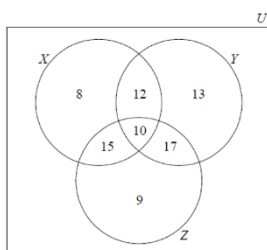
	Smoker	Non-smoker
Male	60	40
Female	30	70

One person from this group is chosen at random.

- Write down the probability that this person is a smoker. $[0.45]$
 - Find the probability that this person is a smoker or is male. $[13/20]$
- 100 students are asked what they had for breakfast on a particular morning. There were three choices: cereal (X), bread (Y) and fruit (Z). It is found that 10 students had all three 17 students had bread and fruit only 15 students had cereal and fruit only 12 students had cereal and bread only 13 students had only bread 8 students had only cereal 9 students had only fruit.
 - Represent this information on a Venn diagram
 - Find the number of students who had none of the three choices for breakfast. $[51\%]$
 - Describe in words what the students in the set had for breakfast
 - Find the probability that a student had at least two of the three choices for breakfast. $[0.54]$
 - Two students are chosen at random. Find the probability that both students had all three choices for breakfast. $[1/110]$



a)

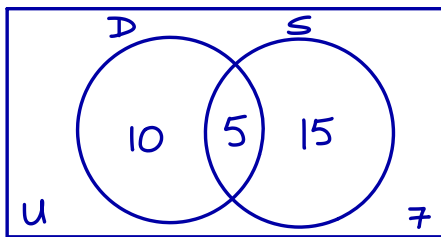


c) (had cereal) and (did not have bread)

Warm-up

In SL Math class, 15 students downhill ski, 20 students snowboard and 5 students do both. There are 37 students. If a student is selected at randomly, state the probability that the student will be:

- a downhill skier?
- does not snowboard or do both?
- a skier given that the student snowboards?



a) The probability that the student will be a downhill skier is $\frac{15}{37}$

$$\begin{aligned} \text{b) } P(S' \cup (D \cap S)) &= P(S') + P(D \cap S) - P(S' \cap (D \cap S)) \\ &= \frac{17}{37} + \frac{5}{37} - 0 \\ &= \frac{22}{37} \end{aligned}$$

The probability that the student does not snowboard or do both is $\frac{22}{37}$

- c) Let A be the event that the student is a skier
Let B be the event that the student snowboards

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{5}{37}}{\frac{20}{37}} \\ &= \frac{5}{20} \\ &= \frac{1}{4} \end{aligned}$$

. The probability is $\frac{1}{4}$

2.4 Conditional Probability and Independent Events

If two events A and B are **mutually exclusive** (both cannot be true at the same time) then

$$P(A \cup B) = P(A) + P(B)$$

$$P(A \cap B) = 0$$

However, when A and B are **not mutually exclusive**, $A \cap B \neq \emptyset$, it can be shown that a more general law applies: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Conditional Probability:

Suppose we have two events A and B , $A|B$ is used to represent that 'A occurs knowing that B has occurred'.

If A and B are events then $P(A|B) = \frac{P(A \cap B)}{P(B)}$ [$P(A|B)$, read as *the probability of A given B*].

It follows that $P(A \cap B) = P(A|B) \cdot P(B)$ or

$$P(A \cap B) = P(B|A) \cdot P(A), \text{ where events } A \text{ and } B \text{ are not mutually exclusive.}$$

Example 1: The probability of rain tomorrow is 30% and the probability of snow tomorrow is 40%. The probability of both rain and snow is 10%.

a) Find the probability of rain or snow tomorrow.

$$P(R) = 0.3, P(S) = 0.4, P(R \cap S) = 0.1$$

$$P(R \cup S) = P(R) + P(S) - P(R \cap S)$$

$$= 0.3 + 0.4 - 0.1$$

$$= 0.6$$

. The probability of rain or snow tomorrow is 60%

b) Find the probability of rain tomorrow if it snows.

$$P(R|S) = \frac{P(R \cap S)}{P(S)}$$

$$= \frac{0.1}{0.4}$$

$$= \frac{1}{4}$$

The probability of rain tomorrow if it snows is 25%

Example 2: The probability that John will go to Western is 0.2 and the probability that he will go to another university is 0.5. If John goes to Western, the probability that his girlfriend Jean will follow him and go to Western is 0.75. What is the probability that John and Jean will both go to Western?

Let A be the event that John will go to Western $P(A) = 0.2$ $P(A') = 0.5$

Let B be the event that Jean will go to Western

$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$\begin{aligned} P(B \cap A) &= P(B|A) \times P(A) \\ &= 0.75 \times 0.2 \\ &= 0.15 \end{aligned}$$

The probability that John and Jean will both go to Western is 0.15

Example 3: Two dice are rolled.

a) Determine the probability that doubles occur or a sum of 8 occurs.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let A be the event that a sum of 8 occurs

Let B be the event that doubles occur

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{36} + \frac{6}{36} - \frac{1}{36}$$

$$= \frac{10}{36}$$

$$= \frac{5}{18}$$

The probability that doubles or a sum of 8 occurs is $\frac{5}{18}$

b) Determine the probability that the sum of the dice is 4 or at least one of the die is showing a 5.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Let A be the event that the sum of the dice is 4

Let B be the event that at least one 5 is showing

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{36} + \frac{11}{36} - \frac{0}{36}$$

$$= \frac{14}{36}$$

$$= \frac{7}{18}$$

The probability that the sum is 4 and at least one 5 is showing is $\frac{7}{18}$

c) What is the probability of rolling a sum greater than 7 with two dice given that the first die rolled is a 3?

1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12
2	3	4	5	6	7	8	9	10	11	12	
3	4	5	6	7	8	9	10	11	12		
4	5	6	7	8	9	10	11	12			
5	6	7	8	9	10	11	12				
6	7	8	9	10	11	12					

Let S be the event of rolling a sum greater than 7
Let F be the event that the 1st die rolled is 3

$$P(S|F) = \frac{P(S \cap F)}{P(F)}$$

$$= \frac{2}{36}$$

$$= \frac{6}{36}$$

$$= \frac{2}{6}$$

$$= \frac{1}{3}$$

The probability of rolling a sum greater than 7 given the 1st die rolled is 3 is $\frac{1}{3}$

Example 4: Five hundred people rate a new television show. One of these people is selected at random.

	A liked it enthusiastically	B liked it moderately	C hated it	D indifferent	Total
male (M)	43	67	91	60	261
female (F)	57	84	28	70	239
Total	100	151	119	130	500

a) What is the probability that the selected person is not a male?

$$P(F) = \frac{239}{500}$$

The probability that the selected person is not a male is $\frac{239}{500}$

b) Find the probability that the selected person is both a male and like the show enthusiastically.

$$P(M \cap A) = \frac{43}{500}$$

The probability that the selected person is both a male and like the show enthusiastically is $\frac{43}{500}$

c) What is the probability that the selected person was a female who liked the show, either moderately or enthusiastically?

$$P(F \cap (A \cup B)) = \frac{57 + 84}{500} = \frac{141}{500}$$

The probability that the selected person was a female who liked the show, either moderately or enthusiastically is $\frac{141}{500}$

d) If it is known that the selected person liked the show moderately, what is the probability the selected person was a male?

$$P(M|B) = \frac{P(M \cap B)}{P(B)} = \frac{\frac{67}{500}}{\frac{151}{500}} = \frac{67}{151}$$

The probability is $\frac{67}{151}$

e) If the selected person is a male, what is the probability that he hated the show?

$$P(C|M) = \frac{P(C \cap M)}{P(M)} = \frac{\frac{91}{500}}{\frac{261}{500}} = \frac{91}{261}$$

The probability is $\frac{91}{261}$

Exit Card!

The following table gives information about Males and Females and whether they prefer Red, Blue, or Yellow.

	Red	Blue	Yellow	Total
Male	20	40	50	110
Female	50	20	20	90
Total	70	60	70	200

Using the table, calculate the following probabilities.

- a) $P(M)$ b) $P(B \cap M)$ c) $P(B|M)$ d) $P(R \cup F)$ e) $P(Y^c \cup F)$

2.4 Practice

- Sue travels the same route to work every day. She has determined that there is a 0.7 probability that she will wait for at least one red light and that there is a 0.4 probability that she will hear her favorite new song on her way to work.
 - What is the probability that she will not have to wait at a red light and will hear her favorite song? [0.12]
 - What is the probability that she will have to wait for a red light and hear her favorite song? [0.28]
- Events A and B have probabilities $P(A) = 0.4$, $P(B) = 0.65$, and $P(A \cup B) = 0.85$.
 - Calculate $P(A \cap B)$.
 - State with a reason whether events A and B are independent.
 - State with a reason whether events A and B are mutually exclusive.
- Given that $P(X) = \frac{2}{3}$, $P(Y|X) = \frac{2}{5}$ and $P(Y|X') = \frac{1}{4}$, find
 - $P(Y')$;
 - $P(X' \cup Y')$.
- The events A and B are such that $P(A) = 0.5$, $P(B) = 0.3$, $P(A \cup B) = 0.6$.
 - (i) Find the value of $P(A \cap B)$.
(ii) Hence show that A and B are not independent.
 - Find the value of $P(B|A)$.
- The events B and C are dependent, where C is the event “a student takes Chemistry”, and B is the event “a student takes Biology”. It is known that $P(C) = 0.4$, $P(B|C) = 0.6$, $P(B|C') = 0.5$.
 - Draw tree diagram.
 - Calculate the probability that a student takes Biology.
 - Given that a student takes Biology, what is the probability that the student takes Chemistry?

Exit Card!

The following table gives information about Males and Females and whether they prefer Red, Blue, or Yellow.

	Red	Blue	Yellow	Total
Male	20	40	50	110
Female	50	20	20	90
Total	70	60	70	200

Using the table, calculate the following probabilities.

a) $P(M)$

b) $P(B \cap M)$

c) $P(B|M)$

d) $P(R \cup F)$

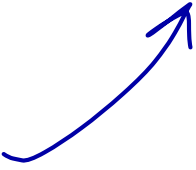
e) $P(Y^c \cup F)$

Recall $Y^c = Y'$

$$\begin{aligned} a) P(M) &= \frac{110}{200} \\ &= \frac{11}{20} \end{aligned}$$

$$\begin{aligned} b) P(B \cap M) &= \frac{40}{200} \\ &= \frac{1}{5} \end{aligned}$$

$$\begin{aligned} c) P(B|M) &= \frac{P(B \cap M)}{P(M)} \\ &= \frac{\frac{40}{200}}{\frac{110}{200}} \end{aligned}$$



$$\begin{aligned} &= \frac{40}{110} \\ &= \frac{4}{11} \end{aligned}$$

$$\begin{aligned} d) P(R \cup F) &= P(R) + P(F) - P(R \cap F) \\ &= \frac{70}{200} + \frac{90}{200} - \frac{50}{200} \\ &= \frac{110}{200} \\ &= \frac{11}{20} \end{aligned}$$

$$\begin{aligned} e) P(Y' \cup F) &= P(Y') + P(F) - P(Y' \cap F) \\ &= \frac{130}{200} + \frac{90}{200} - \frac{70}{200} \\ &= \frac{150}{200} \\ &= \frac{3}{4} \end{aligned}$$

Practice

1. Sue travels the same route to work every day. She has determined that there is a 0.7 probability that she will wait for at least one red light and that there is a 0.4 probability that she will hear her favorite new song on her way to work.
 - a) What is the probability that she will not have to wait at a red light and will hear her favorite song? [0.12]

Let A represent the event of sue waiting a t a red light

Let B represent the event of Sue hearing her favorite song.

a)

$$\begin{aligned}P(A' \cap B) &= P(A') \times P(B) \\&= [1 - P(A)] \times P(B) \\&= [1 - 0.7] \times 0.4 \\&= 0.1\end{aligned}$$

There is a 12% probability

- b) What is the probability that she will have to wait for a red light and hear her favorite song?[0.28]

$$\begin{aligned}P(A \cap B) &= P(A) \times P(B) \\&= 0.7 \times 0.4 \\&= 0.28\end{aligned}$$

There is a 28% probability.

2. Events A and B have probabilities $P(A) = 0.4$, $P(B) = 0.65$, and $P(A \cup B) = 0.85$.
 - a) Calculate $P(A \cap B)$.

$$\begin{aligned}P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\&= 0.4 + 0.65 - 0.85 \\&= 0.2\end{aligned}$$

- b) State with a reason whether events A and B are independent.

Event A and B are not independent since

$$\begin{aligned}P(A \cap B) &\neq P(A) \times P(B) \\0.2 &\neq 0.4 \times 0.65 \\0.2 &\neq 0.26\end{aligned}$$

- c) State with a reason whether events A and B are mutually exclusive.

Event A and B are not mutually exclusive since

$$P(A \cap B) \neq 0$$

3. Given that $P(X) = \frac{2}{3}$, $P(Y|X) = \frac{2}{5}$ and $P(Y|X') = \frac{1}{4}$, find

a) $P(Y')$

$$\textcircled{1} \quad P(Y|X) = \frac{2}{5}$$

$$\frac{P(Y \cap X)}{P(X)} = \frac{2}{5}$$

$$\begin{aligned} P(Y \cap X) &= \frac{2}{5} P(X) \\ &= \frac{2}{5} \times \frac{2}{3} \\ &= \frac{4}{15} \end{aligned}$$

$$\textcircled{2} \quad P(Y|X') = \frac{1}{4}$$

$$\frac{P(Y \cap X')}{P(X')} = \frac{1}{4}$$

$$\begin{aligned} P(Y \cap X') &= \frac{1}{4} P(X') \\ &= \frac{1}{4} \left[1 - \frac{2}{3} \right] \\ &= \frac{1}{12} \end{aligned}$$

$$\textcircled{3} \quad P(Y \cap X') = P(Y) - P(Y \cap X)$$

$$\frac{1}{12} = P(Y) - \frac{4}{15}$$

$$\begin{aligned} P(Y) &= \frac{1}{12} + \frac{4}{15} \\ &= \frac{7}{20} \end{aligned}$$

$$\therefore P(Y') = 1 - \frac{7}{20} = \frac{13}{20}$$

b) $P(X' \cup Y')$

$$\begin{aligned} P(X' \cup Y') &= P(X \cap Y)' \\ &= 1 - P(X \cap Y) \\ &= 1 - \frac{4}{15} \\ &= \frac{11}{15} \end{aligned}$$

4. The events A and B are such that $P(A) = 0.5$, $P(B) = 0.3$, $P(A \cup B) = 0.6$.

a) (i) Find the value of $P(A \cap B)$.

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.5 + 0.3 - 0.6 \\ &= 0.2 \end{aligned}$$

(ii) Hence show that A and B are not independent.

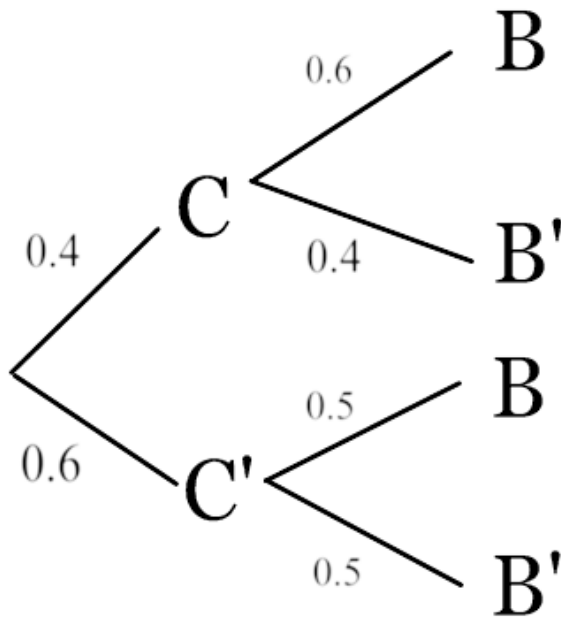
$$\begin{aligned} P(A \cap B) &\neq P(A) \times P(B) \\ 0.2 &\neq 0.5 \times 0.3 \end{aligned}$$

b) Find the value of $P(B|A)$.

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{0.2}{0.5} \\ &= \frac{2}{5} \end{aligned}$$

5. The events B and C are dependent, where C is the event “a student takes Chemistry”, and B is the event “a student takes Biology”. It is known that $P(C) = 0.4$, $P(B|C) = 0.6$, $P(B|C') = 0.5$.
- Draw tree diagram.
 - Calculate the probability that a student takes Biology.
 - Given that a student takes Biology, what is the probability that the student takes Chemistry?

a)



b)

$$\begin{aligned} P(B) &= (0.4)(0.6) + (0.6)(0.5) \\ &= 0.54 \end{aligned}$$

c)

$$\begin{aligned} P(C|B) &= \frac{P(C \cap B)}{P(B)} \\ &= \frac{(0.4)(0.6)}{0.54} \\ &= 0.44 \end{aligned}$$

Warm-Up

1. In a survey, 100 students were asked „do you prefer to watch television or play sport?“ Of the 46 boys in the survey, 33 said they would choose sport, while 29 girls made this choice.

	Boys (B)	Girls	Total
Television (T)	13	25	38
Sport	33	29	62
Total	46	54	100

By completing this table or otherwise, find the probability that

- a) a student selected at random prefers to watch television.

$$P(T) = \frac{38}{100} = \frac{19}{50} \quad \text{The probability is } \frac{19}{50}$$

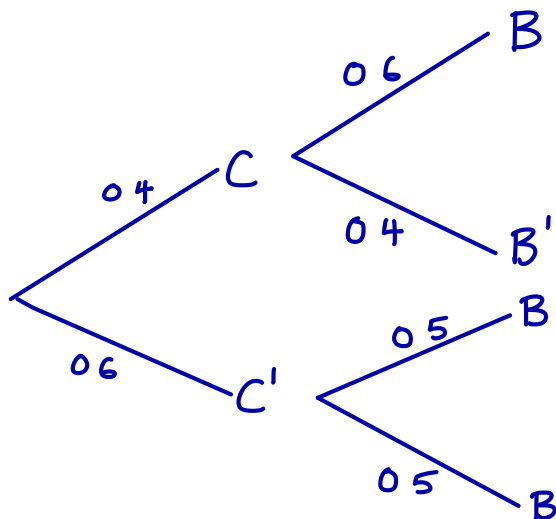
- b) a student prefers to watch television, given that the student is a boy.

$$P(T|B) = \frac{P(T \cap B)}{P(B)} = \frac{\frac{13}{100}}{\frac{46}{100}} = \frac{13}{46} \quad \text{The probability is } \frac{13}{46}$$

2. The events B and C are dependent, where C is the event "a student takes Chemistry", and B is the event "a student takes Biology". It is known that

$$P(C) = 0.4, P(B|C) = 0.6, P(B|C') = 0.5.$$

Create a tree diagram to represent the sample space.



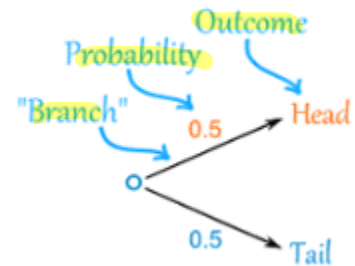
2.5 Probability Tree Diagrams (Independent Events):

Calculating probabilities can be hard, sometimes we add them, sometimes we multiply them, and often it is hard to figure out what to do ... tree diagrams can help

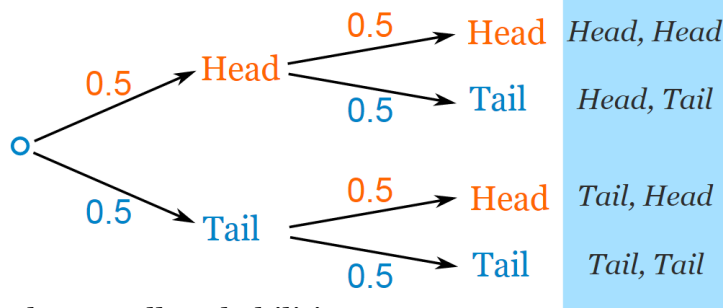
Example: Here is a tree diagram for the toss of a coin:

There are two "branches" (Heads and Tails)

- The probability of each branch is written on the branch
- The outcome is written at the end of the branch

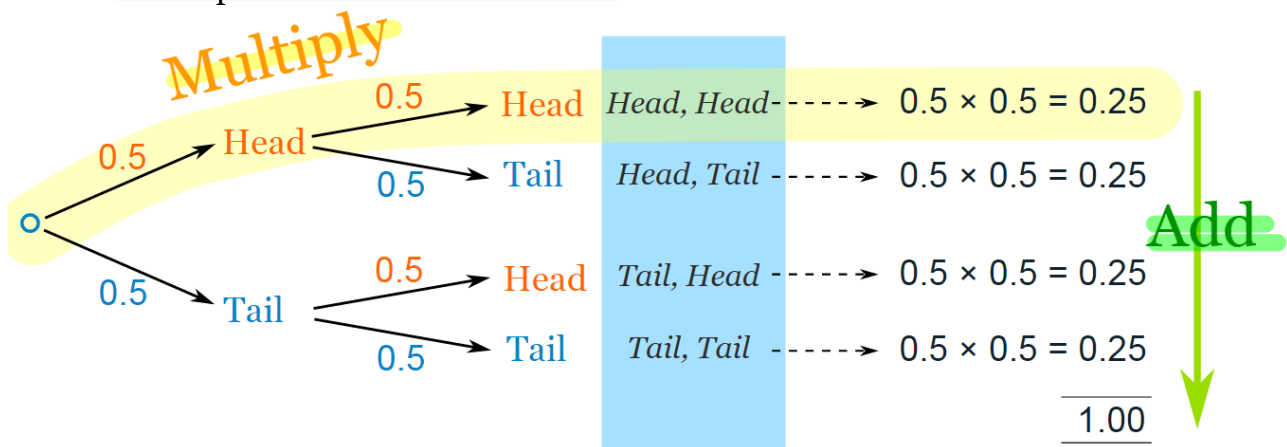


Example: We can extend the tree diagram to two tosses of a coin:



How do we calculate the overall probabilities?

- We **multiply** probabilities **along the branches**
- We **add** probabilities **down columns**



Multiplication Law for Conditional Probability:

I: Suppose we have two **Independent** Events A and B , where the occurrence of either event **does not** affect the probability of the other, then:

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

II: Suppose we have two **Dependent** events A and B , where the occurrence of either event **does** affect the probability of the other, then:

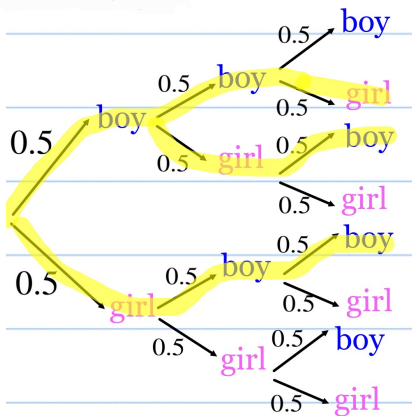
$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$P(A \text{ then } B) = P(A) \cdot P(B \text{ given that } A \text{ has occurred})$$

Example 4: In a three-child family, assuming equal chance of giving birth to a boy or a girl, what is the probability of

a) having two boys?

b) having at least one girl?



$$\begin{aligned} a) P(2 \text{ boys}) &= P(BBG) + P(BGB) + P(GBB) \\ &= \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \end{aligned}$$

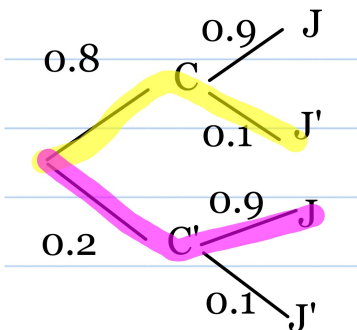
$$= \frac{3}{8}$$

- The probability of having 2 boys is $\frac{3}{8}$

$$\begin{aligned} b) P(\text{at least 1 girl}) &= 1 - P(BBB) \\ &= 1 - \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) \\ &= \frac{7}{8} \end{aligned}$$

- The probability of having at least 1 girl is $\frac{7}{8}$

Example 5: In Archery, Charlie hits the target 80% of the time and Jim hits the target 90% of the time. If Charlie shoots and then Jim, what is the probability that the target will get hit exactly once?



$$\begin{aligned} P(\text{one hit}) &= P(CJ') + P(C'J) \\ &= (0.8 \times 0.1) + (0.2 \times 0.9) \\ &= 0.08 + 0.18 \\ &= 0.26 \end{aligned}$$

- The probability that the target will get hit exactly once is 26%

Example 6: Cal is not having much luck lately. His car will only start 80% of the time and his motorbike will only start 60% of the time

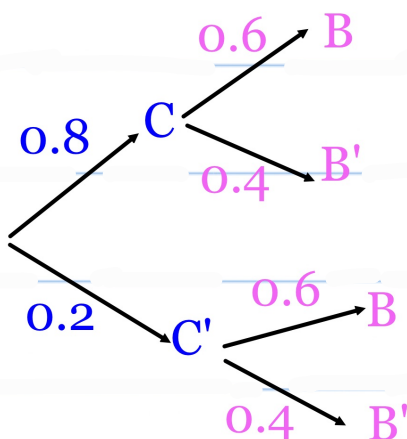
a) Draw a tree diagram to illustrate this situation

b) Use the tree diagram to determine the chance that

i) both will start

ii) Carl can only use his car.

Let C be the event that the car starts
Let B be the event that the bike starts



$$\begin{aligned} b) i) P(C \cap B) &= 0.8 \times 0.6 \\ &= 0.48 \end{aligned}$$

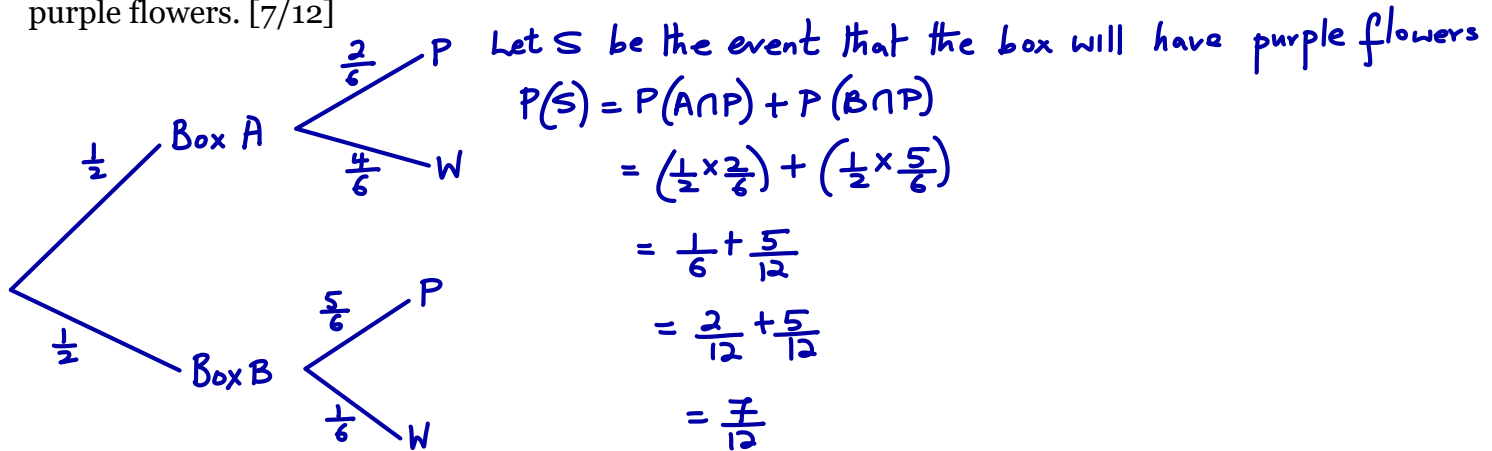
The probability that both will start is 0.48

$$\begin{aligned} ii) P(C') &= 0.8 \times 0.4 \\ &= 0.32 \end{aligned}$$

- The probability that Carl can only use his car is 0.32

You Try!

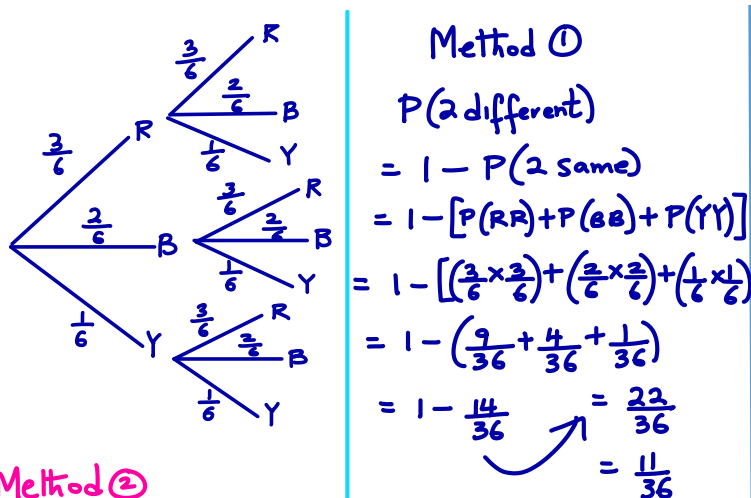
1) Two boxes each contain 6 petunia plants that are not yet flowering. Box A contains 2 plants that will have purple flowers and 4 plants that will have white flowers. Box B contains 5 plants that will have purple flowers and 1 plant that will have white flowers. A box is selected by tossing a coin, and one plant is removed at random from it. Determine the probability that it will have purple flowers. [7/12]



The probability that the box will have purple flowers is $\frac{7}{12}$

2) A box contains 3 red, 2 blue and 1 yellow marble. Find the probability of getting two different color

a) if replacement occurs [11/18] b) if replacement does not occur. [11/15]



Method ②

$$P(2 \text{ different colours})$$

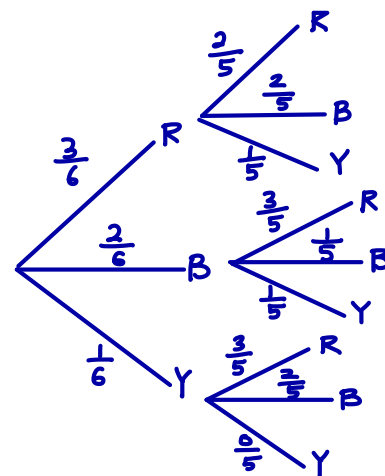
$$= P(RB) + P(RY) + P(BR) + P(BY) + P(YR) + P(YB)$$

$$= \left(\frac{3}{6} \times \frac{2}{6}\right) + \left(\frac{3}{6} \times \frac{1}{6}\right) + \left(\frac{2}{6} \times \frac{3}{6}\right) + \left(\frac{2}{6} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{3}{6}\right) + \left(\frac{1}{6} \times \frac{2}{6}\right)$$

$$= \frac{6}{36} + \frac{3}{36} + \frac{6}{36} + \frac{2}{36} + \frac{3}{36} + \frac{2}{36}$$

$$= \frac{22}{36} \rightarrow \frac{11}{18}$$

The probability is $\frac{11}{18}$



$$P(2 \text{ different})$$

$$= 1 - P(2 \text{ same})$$

$$= 1 - [P(RR) + P(BB) + P(YY)]$$

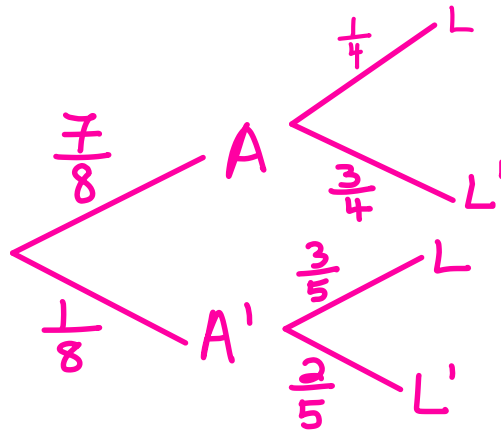
$$= 1 - \left[\left(\frac{3}{6} \times \frac{2}{5}\right) + \left(\frac{2}{6} \times \frac{1}{5}\right) + \left(\frac{1}{6} \times \frac{0}{5}\right)\right]$$

$$= 1 - \left(\frac{6}{30} + \frac{2}{30}\right) = 1 - \frac{8}{30} = \frac{22}{30} = \frac{11}{15}$$

The probability is $\frac{11}{15}$

Exit Card!

Use a tree diagram to help you solve the following problem: On any given day, the probability that Rachel will be woken by her alarm clock is $\frac{7}{8}$. If she is woken by her alarm clock, the probability that she will be late for school is $\frac{1}{4}$. If she is not woken by her alarm clock, the probability she will be late for school is $\frac{3}{5}$. Calculate the probability that Rachel will arrive on time for school.



$$\begin{aligned} P(\text{not late}) &= P(A \cap L') + P(A' \cap L') \\ &= \left(\frac{7}{8} \times \frac{3}{4}\right) + \left(\frac{1}{8} \times \frac{2}{5}\right) \\ &= \frac{21}{32} + \frac{1}{20} \\ &= \frac{105+8}{160} \\ &= \frac{113}{160} \end{aligned}$$

. The probability that Rachel will arrive on time is $\frac{113}{160}$

2.5 Practice

1. Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.
 - a) Write down the probability that Jack wins on his first throw.
 - b) Calculate the probability that Jill wins on her first throw.
 - c) Calculate the probability that Jack wins the game.

2. A bag contains 10 red balls, 10 green balls and 6 white balls. Two balls are drawn at random from the bag without replacement. What is the probability that they are of different colours?

3. Bag A contains 2 red and 3 green balls.
 - a) Two balls are chosen at random from the bag without replacement. Find the probability that 2 red balls are chosen.Bag B contains 4 red and n green balls
 - b) Two balls are chosen without replacement from this bag. If the probability that two red balls are $\frac{2}{15}$, show that $n=6$.A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.
 - c) Calculate the probability that two red balls are chosen.
 - d) Given that two red balls are chosen, find the probability that a 1 or a 6 was obtained on the die.

4. Box A contains 6 red balls and 2 green balls. Box B contains 4 red balls and 3 green balls. A fair cubical die with faces numbered 1, 2, 3, 4, 5, 6 is thrown. If an even number is obtained, a ball is selected from box A; if an odd number is obtained, a ball is selected from box B.
 - a) Calculate the probability that the ball selected was red.
 - b) Given that the ball selected was red, calculate the probability that it came from box B.

Practice

1. Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.
 - a) Write down the probability that Jack wins on his first throw.
 - b) Calculate the probability that Jill wins on her first throw.
 - c) Calculate the probability that Jack wins the game.

(a) $\frac{2}{3}$, since 4 out of 6 opening rolls wins.

$$(b) P(\text{Jill wins on the 1st throw}) = P(\text{Jack gets 5 or 6}) * P(\text{Jill gets 1, -4}) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$$

(c) Let X be the probability that the person who rolls first wins. Then 1-X is the probability that the person who rolls second wins. Notice that X is the sum of the probability that Jack wins on the first roll plus the probability that he wins on a later roll.

Since the first part of this was determined in (a), we must only determine the probability that Jack wins on a later roll. In order for this to occur, he must roll a 5 or 6, followed by Jill taking a turn. When Jill takes her turn, her probability of winning is now X (since in essence, she is the first player in the game), and her probability of losing is 1-X. Thus the

probability of Jack winning is this product. This leads us to the equation:

$$X = \frac{2}{3} + \frac{1}{3}(1 - X)$$

$$X = 1 - \frac{X}{3}$$

$$X = \frac{3}{4}$$

2. A bag contains 10 red balls, 10 green balls and 6 white balls. Two balls are drawn at random from the bag without replacement. What is the probability that they are of different colours?

$$P(\text{different colours}) = 1 - [P(GG) + P(RR) + P(WW)]$$

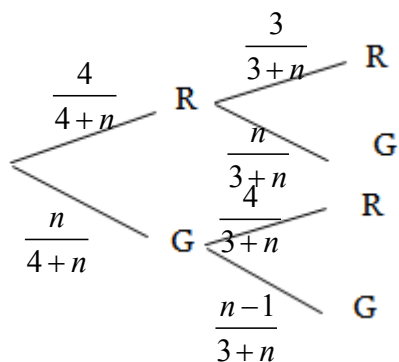
$$= 1 - \left(\frac{10}{6} \times \frac{9}{25} + \frac{10}{26} \times \frac{9}{25} + \frac{6}{26} \times \frac{5}{25} \right) = 1 - \left(\frac{210}{650} \right) = \frac{44}{65} (= 0.677, \text{ to 3 sf})$$

3. Bag A contains 2 red and 3 green balls.
a) Two balls are chosen at random from the bag without replacement. Find the probability that 2 red balls are chosen.

$$P(2\text{Red}) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$$

Bag B contains 4 red and n green balls

- b) Two balls are chosen without replacement from this bag. If the probability that two red balls are chosen is $\frac{2}{15}$, show that $n=6$.



$$\frac{4}{4+n} \times \frac{3}{3+n} = \frac{2}{15}$$

$$\frac{12}{n^2 + 7n + 12} = \frac{2}{15}$$

$$n^2 + 7n + 12 = 90$$

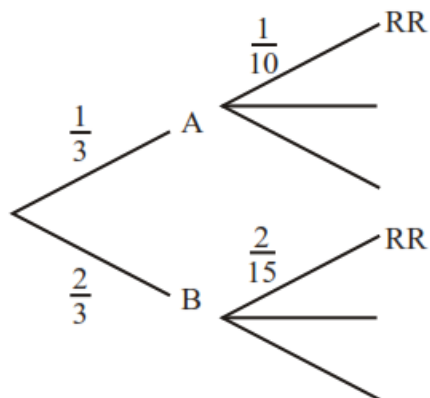
$$n^2 + 7n - 78 = 0$$

$$(n+13)(n-6) = 0$$

$$n = -13, \boxed{n = 6}$$

A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.

- c) Calculate the probability that two red balls are chosen.



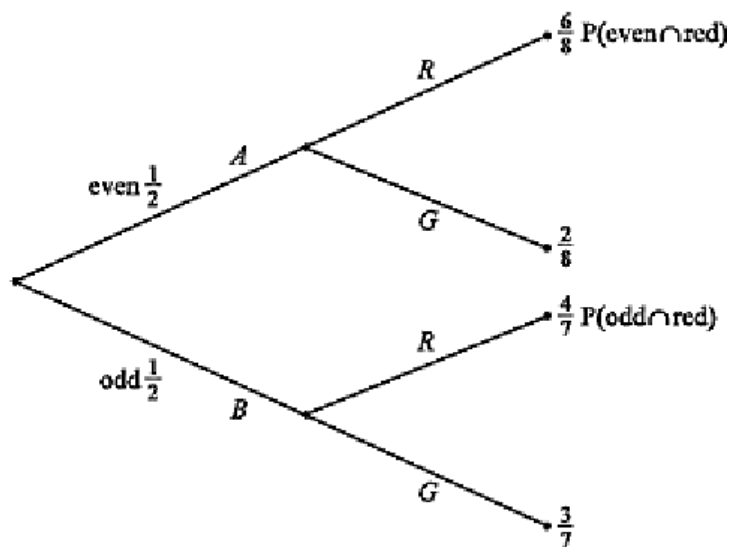
$$P(RR) = \frac{1}{3} \times \frac{1}{10} + \frac{2}{3} \times \frac{2}{15} = \frac{11}{90}$$

- d) Given that two red balls are chosen, find the probability that a 1 or a 6 was obtained on the die.

$$P(1 \text{ or } 6) = P(A)$$

$$P(A|RR) = \frac{P(A \cap RR)}{P(RR)} = \frac{\left[\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)\right]}{\frac{11}{90}} = \frac{3}{11}$$

4. Box A contains 6 red balls and 2 green balls. Box B contains 4 red balls and 3 green balls. A fair cubical die with faces numbered 1, 2, 3, 4, 5, 6 is thrown. If an even number is obtained, a ball is selected from box A; if an odd number is obtained, a ball is selected from box B.
- Calculate the probability that the ball selected was red.
 - Given that the ball selected was red, calculate the probability that it came from box B.



$$\begin{aligned} \text{(a)} \quad P(R) &= \frac{1}{2} \times \frac{6}{8} + \frac{1}{2} \times \frac{4}{7} \\ &= \frac{3}{8} + \frac{2}{7} = \frac{37}{56} \quad (0.661) \end{aligned}$$

$$\text{(b)} \quad P(B|R) = \frac{P(B \cap R)}{P(R)} = \frac{\frac{2}{7}}{\frac{37}{56}} = \frac{16}{37} \quad (0.432)$$

2.6 Probability Distribution

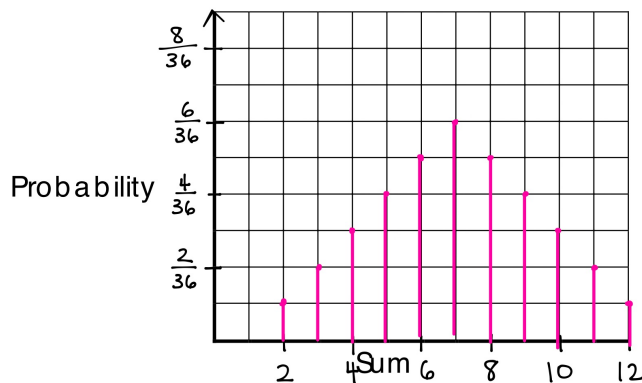
Example 1: Determine the probability distribution for the sum of the roll of two dice.

		1 st die					
		1	2	3	4	5	6
2 nd die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Tabulate the probability distribution:

		x										
	Sum X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Note: Sum of probabilities is 1. $\sum P(X=x) = 1$



To construct a probability model, it is necessary to assign a numerical value to each outcome. This assignment of a numerical value to a real-life occurrence is called the **random variable** and is denoted by X . Example: X was the sum of the two dice. X defines the outcome

A random variable, X , is a measurable quantity which can take any value. Its value is the result of a random observation or experiment. Actual measured values are represented by x . x is the actual value of the outcome

Random variables with outcomes that are assigned **integral values** (i.e. 0, 1, 2, ...) or certain discrete values are called **discrete random variables**.

Associated with each value of the random variable will be a probability of the value occurring, denoted by $P(X=x)$ which is called the **probability distribution**.

$\sum P(X=x)=1$, Sum of the probabilities of all of the outcomes must equal one.

Example: Determine the probability distribution for the tossing of a coin.

Solution: Let the random variable X be 'the number of tail obtained'

#Tails $\rightarrow x$	0	1
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

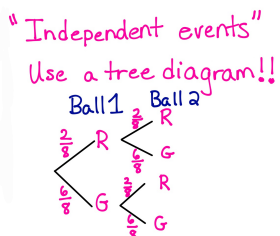
$P(X=0)$ $P(X=1)$

Note: $P(X=x)$ is the probability that the number of tail is x .

Example 1: For each of these examples, draw a table of possible values of x together with the associated probability $P(X=x)$

a) A box contains 2 red balls and 6 green balls. Two balls are chosen at random with replacement, and X is the number of green balls obtained.

Solution:



Ball 1	Ball 2	# of green balls (x)	Probability
R	R	0	$\frac{2}{8} \times \frac{2}{8} = \frac{1}{16}$
R	G	1	$\frac{2}{8} \times \frac{6}{8} = \frac{3}{16}$
G	R	1	$\frac{6}{8} \times \frac{2}{8} = \frac{3}{16}$
G	G	2	$\frac{6}{8} \times \frac{6}{8} = \frac{9}{16}$

The probability distribution for X is:

x	0	1	2
$P(X=x)$	$\frac{1}{16}$	$\frac{3}{16} + \frac{3}{16} = \frac{6}{16}$	$\frac{9}{16}$

$$\sum P(X=x) = 1$$

b) A fair die has faces labelled 1, 1, 1, 2, 3, 3 and X is the score when the die is thrown.

Solution: The probability distribution for X is:

x	1	2	3
$P(X=x)$	$\frac{3}{6} = \frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{6} = \frac{1}{3}$

c) Two fair dice are thrown and X is the difference between the higher score and the lower score.

Solution:

1st die

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

2nd die

The probability distribution for X is:

x	0	1	2	3	4	5
$P(X=x)$	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$
$P(X=x)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

Expectation: The **expected value** is that quantity that you can expect to obtain when the experiment is performed or the **mean value** of the random variable X and is denoted by $E(X)$. The expected value of X is $E(X) = \sum x P(X=x)$ $E(X) = \mu = x_1 p_1 + x_2 p_2 + \dots + x_k p_k$

Note: The value of each outcome is multiplied by the probability of its occurrence and then these values are added to obtain the expected value of the experiment.

Example 2: A fair coin is thrown twice and X is the number of tails obtained. Find $E(X)$.

Solution:

1 st throw	2 nd throw	# of tails (x)	Probability
T	T	2	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
T	H	1	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
H	T	1	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
H	H	0	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

The probability distribution for X is:

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E(X) = \sum x P(X=x)$$

$$= 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right)$$

$$= 1$$

The expected number of tails is 1

Example 3: The random variable X can only take the values 1, 2 and 3. Given that the value 3 is twice as likely as each of the values 1 and 2, and values 1 and 2 are equally likely,

- draw a table of possible values of x together with $P(X=x)$,
- determine the expectation of X .

a) Let $P(X=1)$ and $P(X=2)$ equal to p , then $P(X=3) = 2p$

x	1	2	3
$P(X=x)$	p	p	$2p$

$$p + p + 2p = 1$$

$$4p = 1$$

$$p = \frac{1}{4}$$

a)

x	1	2	3
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

b) $E(X) = \sum x P(X=x)$

$$E(X) = 1\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{2}\right)$$

$$= \frac{9}{4}$$

Example 4: The random variable X has a probability distribution given by:

$P(X=x) = \frac{x}{k}$, $x = 1, 2, 3, 4$ (a) Find the value of the constant k . (b) Calculate $E(X)$.

a)

x	1	2	3	4
$P(X=x)$	$\frac{1}{k}$	$\frac{2}{k}$	$\frac{3}{k}$	$\frac{4}{k}$

$$\therefore \frac{1}{k} + \frac{2}{k} + \frac{3}{k} + \frac{4}{k} = 1$$

$$\frac{10}{k} = 1$$

$$k = 10$$

b) $E(X) = \sum x P(X=x)$

$$= x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4$$

$$E(X) = 1\left(\frac{1}{10}\right) + 2\left(\frac{2}{10}\right) + 3\left(\frac{3}{10}\right) + 4\left(\frac{4}{10}\right)$$

$$= \frac{1}{10} + \frac{4}{10} + \frac{9}{10} + \frac{16}{10}$$

$$= 3$$

Example: A game is defined by the rules that 2 dice are rolled and the player wins varying amounts depending on the sum of the 2 dice according to the following table:

Sum	2	3	4	5	6	7	8	9	10	11	12
Winnings (\$)	10	9	8	7	6	5	6	7	8	9	10

The cost to play the game is \$7.50.

- What can a player expect to win by playing this game?
- What would be a fair value to pay to play this game?

a) Let the random variable X be the winnings associated with each roll

Winnings(\$), X	Sum	Probabilities, $P(X=x)$
5	7	6/36
6	6,8	10/36
7	5,9	8/36
8	4,10	6/36
9	3,11	4/36
10	2,12	2/36

Die 1		1	2	3	4	5	6
Die 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12



The expected value of the defined game is:

$$\begin{aligned}
 E(X) &= \sum x P(X=x) \\
 &= 5\left(\frac{6}{36}\right) + 6\left(\frac{10}{36}\right) + 7\left(\frac{8}{36}\right) + 8\left(\frac{6}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{2}{36}\right) \\
 &= \frac{30}{36} + \frac{60}{36} + \frac{56}{36} + \frac{48}{36} + \frac{36}{36} + \frac{20}{36} \\
 &= \frac{250}{36} \\
 &\doteq 6.94
 \end{aligned}$$

$\therefore$ the expected winning is \$6.94.

b) If you pay \$7.50 per game, then you would expect to lose $\$7.50 - \$6.94 = \$0.56$ per game in the long run. This would be a fair game only if the cost to play was \$6.94

2.6 Practice

1. If $P(X = 0) = \alpha$, $P(X = 1) = 2\alpha$, $P(X = 3) = 3\alpha$, find α .

2. The probability distribution of a random variable X is represented by the function

$$P(X = x) = \frac{k}{x}, \text{ where } x = 1, 2, 3, 4, 5, 6. \text{ Find a) the value of } k \quad \text{b) } P(3 \leq X \leq 5)$$

3. A discrete random variable X has a probability distribution defined by the function:

$$P(X = x) = \binom{4}{x} \left(\frac{2}{5}\right)^x \left(\frac{3}{5}\right)^{4-x} \text{ where } x = 0, 1, 2, 3, 4.$$

a. Display the function using a table and graph b. Find $P(X = 2)$ c. $P(1 \leq X \leq 3)$

4. You draw one card from a standard deck of playing cards. If you pick a heart, you will win \$10. If you pick a face card, which is not a heart, you win \$8. If you pick any other card, you lose \$6. Do you want to play? Explain.

5. The world famous gambler from Philadelphia, Señor Rick, proposes the following game of chance. You roll a fair die. If you roll a 1, then Señor Rick pays you \$25. If you roll a 2, Señor Rick pays you \$5. If you roll a 3, you win nothing. If you roll a 4 or a 5, you must pay Señor Rick \$10, and if you roll a 6, you must pay Señor Rick \$15. Is Señor Rick loco for proposing such a game? Explain.

6. You pay \$10 to play the following game of chance. There is a bag containing 12 balls, five are red, three are green and the rest are yellow. You are to draw one ball from the bag. You will win \$14 if you draw a red ball and you will win \$12 if you draw a yellow ball. How much do you expect to win or lose if you play this game 100 times?

7. A detective figures that he has a one in nine chance of recovering stolen property. His out-of-pocket expenses for the investigation are \$9,000. If he is paid his fee only if he recovers the stolen property, what should he charge clients in order to breakeven?

8. At Tucson Raceway Park, your horse, Soon-to-be-Glue, has a probability of $1/20$ of coming in first place, a probability of $1/10$ of coming in second place, and a probability of $1/4$ of coming in third place. First place pays \$4,500 to the winner, second place \$3,500 and third place \$1,500. Is it worthwhile to enter the race if it costs \$1,000?

9. Your company plans to invest in a particular project. There is a 35% chance that you will lose \$30,000, a 40% chance that you will break even, and a 25% chance that you will make \$55,000. Based solely on this information, what should you do?

10. A manufacturer is considering the manufacture of a new and better mousetrap. She estimates the probability that the new mousetrap is successful is $3/4$. If it is successful it would generate profits of \$120,000. The development costs for the mousetrap are \$98,000. Should the manufacturer proceed with plans for the new mousetrap? Why or why not?

Answers:

1. $\alpha + 2\alpha + 3\alpha = 1$

$$6\alpha = 1$$

$$\alpha = \frac{1}{6}$$

2. a) $\frac{k}{1} + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} + \frac{k}{5} + \frac{k}{6} = 1$

$$k \left(\frac{49}{20} \right) = 1$$

$$k = \frac{20}{49}$$

b) $P(X = x) = \frac{20}{49x}$

$$P(3 \leq X \leq 5) = P(X = 3) + P(X = 4) + P(X = 5)$$

$$= \frac{20}{49(3)} + \frac{20}{49(4)} + \frac{20}{49(5)}$$

$$= \frac{47}{147}$$

3. a) $P(X = 0) = \left(\frac{3}{5} \right)^4 = \frac{81}{625}$

$$P(X = 1) = 4 \left(\frac{2}{5} \right) \left(\frac{3}{5} \right)^3 = \frac{216}{625}$$

$$P(X = 2) = 6 \left(\frac{2}{5} \right)^2 \left(\frac{3}{5} \right)^2 = \frac{216}{625}$$

$$P(X = 3) = 4 \left(\frac{2}{5} \right)^3 \left(\frac{3}{5} \right) = \frac{96}{625}$$

$$P(X = 4) = \left(\frac{2}{5} \right)^4 = \frac{16}{625}$$

b) **$P(X = 2) = 6 \left(\frac{2}{5} \right)^2 \left(\frac{3}{5} \right)^2 = \frac{216}{625}$**

c) $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$

$$= \frac{216}{625} + \frac{216}{625} + \frac{96}{625}$$

$$= \frac{528}{625}$$

4. Let X be the random variable that takes on the values 10, 8 and -6 , the values of the winnings. First, we calculate the following probabilities:

$$P(X = 10) = \frac{13}{52}, P(X = 8) = \frac{9}{52}, \text{ and } P(X = -6) = \frac{30}{52}.$$

The expected value of the game is

$$\begin{aligned} E(X) &= P(X = 10) * 10 + P(X = 8) * 8 - P(X = -6) * 6 \\ &= \frac{13}{52} * 10 + \frac{9}{52} * 8 - \frac{30}{52} * 6 \\ &= \frac{130 + 72 - 180}{52} \\ &= \frac{22}{52} \end{aligned}$$

Since the expected value of the game is approximately \$.42, it is to the player's advantage to play the game.

5. This is very similar to the first problem. Let X be the random variable take takes on the values 25, 5, 0, -10, -15, the values of the winnings. A simple calculation yields the following probabilities:

$$P(X = 25) = \frac{1}{6}, P(X = 5) = \frac{1}{6}, P(X = 3) = \frac{1}{6}, P(X = -10) = \frac{2}{6}, \text{ and } P(X = -15) = \frac{1}{6}.$$

The expected value is given by

$$\begin{aligned} E(X) &= 25 * \frac{1}{6} + 5 * \frac{1}{6} + 0 * \frac{1}{6} - 10 * \frac{2}{6} - 15 * \frac{1}{6} \\ &= \frac{25 + 5 + 0 - 20 - 15}{6} \\ &= -\frac{5}{6} \end{aligned}$$

Therefore, Señor Rick is not loco since the expected value is approximately -.83.

6. Here, the gross winnings are 14, 12, or 0. Since you must pay \$10 to play, the net winnings are 4, 2, and -10. Let X be the random variable that takes on the values 4, 2, and -10, the values of the net winnings.

$$P(X = 4) = \frac{5}{12}, \quad P(X = 2) = \frac{4}{12}, \quad \text{and} \quad P(X = -10) = \frac{3}{12}.$$

The expected value of the game is given by

$$\begin{aligned} E(X) &= 4 * \frac{5}{12} + 2 * \frac{4}{12} - 10 * \frac{3}{12} \\ &= \frac{20 + 8 - 30}{12} \\ &= -\frac{2}{12} \end{aligned}$$

You should expect to lose \$16.67 after one-hundred games.

7. In this problem we want to determine the detective's fee so that the expected value is zero.

Let y be the amount of his fee. Let X be the random variable that takes on the values y or 0, the amount he charges for a job. Then $P(X = y) = \frac{1}{9}$ and $P(X = 0) = \frac{8}{9}$.

The detective is out \$9,000 regardless of whether he recovers the stolen property. So we have

$$\begin{aligned} E(X) &= (y - 9000) * P(X = y) - 9000 * P(X = 0) \\ &= (y - 9000) * \frac{1}{9} - 9000 * \frac{8}{9} \\ &= 0 \end{aligned}$$

Solving for y we see that the detective must charge \$81,000.

8. The gross winnings are \$4,500, \$3,500 and \$1,500. The net winnings are \$3,500, \$2,500, \$500 and -\$1,000. Let X be the random variable that takes on the value of the net winnings.

Then $P(X = 3500) = \frac{1}{20}$, $P(X = 2500) = \frac{1}{10}$, $P(X = 500) = \frac{1}{4}$, and $P(X = -1000) = \frac{12}{20}$.

$$\begin{aligned} E(X) &= 3500 * \frac{1}{20} + 2500 * \frac{2}{20} + 500 * \frac{5}{20} - 1000 * \frac{12}{20} \\ &= \frac{3500 + 5000 + 2500 - 12000}{20} \\ &= -50 \end{aligned}$$

So it would appear that Soon-to-be-Glue is soon to be glue.

9. Let X be the random variable that takes on takes on the value of the investment.
 $P(X = -30000) = 0.35$., $P(X = 0) = 0.4$ and $P(X = 55000) = 0.25$.

The expected value of the project is

$$\begin{aligned} E(X) &= -30000 * 0.35 + 0 * 0.4 + 55000 * .25 \\ &= \$3,250 \end{aligned}$$

Since the expected value is positive, you should proceed with the project.

10. A manufacturer is considering the manufacture of a new and better mousetrap. She estimates the probability that the new mousetrap is successful is 0.75. If it is successful it would generate profits of \$120,000. The development costs for the mousetrap are \$98,000. Should the manufacturer proceed with plans for the new mousetrap? Why or why not?

expected value = $.75(120000) + .25(-98000) = \$65,500$
 Expected profit of 65,500, proceed with the project.

Warm Up

Allen and Emily, each throw two fair cubical dice simultaneously. The score for each player is the sum of the two numbers shown on their respective dice.

(a) (i) Calculate the probability that Allen obtains a score of 9.

(ii) Calculate the probability that Allen and Emily both obtain a score of 9.

(b) (i) Calculate the probability that Allen and Emily obtain the same score.

(ii) Deduce the probability that Allen's score exceeds Emily's score.

(c) Let X denote the largest number shown on the four dice.

(i) Show that $P(X \leq x) = \left(\frac{x}{6}\right)^4$, for $x = 1, 2, 3, 4, 5, 6$

(ii) Copy and complete the following probability distribution table.

x	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$				$\frac{671}{1296}$

(iii) Calculate $E(X)$.

Solution

(a) (i) Calculate the probability that Allen obtains a score of 9.

$\frac{4}{36} = \frac{1}{9}$, since the rolls (3,6), (4,5), (5,4) and (6,3) sum to 9 out of the 36 possible rolls.

(ii) Calculate the probability that Allen and Emily both obtain a score of 9.

$$P(A = 9 \cap E = 9) = \frac{1}{9} \times \frac{1}{9} = \frac{1}{81}, \text{ since their rolls are independent.}$$

(b) (i) This is tedious. In general, if the probability of obtaining some score is k , the probability both of them obtain it is k^2 . So, we must sum over all 11 possible scores.

$$\begin{aligned} P(A = E) &= \left(\frac{1}{36}\right)^2 + \left(\frac{2}{36}\right)^2 + \left(\frac{3}{36}\right)^2 + \left(\frac{4}{36}\right)^2 + \left(\frac{5}{36}\right)^2 + \left(\frac{6}{36}\right)^2 + \left(\frac{5}{36}\right)^2 + \left(\frac{4}{36}\right)^2 + \left(\frac{3}{36}\right)^2 + \left(\frac{2}{36}\right)^2 + \left(\frac{1}{36}\right)^2 \\ &= \frac{146}{1296} \\ &= \frac{73}{648} \end{aligned}$$

$$(ii) P(A > E) = P(E > A), \text{ so } P(A > E) = \frac{1 - P(A = E)}{2} = \frac{1 - \frac{73}{648}}{2} = \frac{575}{1296}$$

(c) (i) The probability of obtain a score of x or less is $\frac{x}{6}$, since there are x choices out of six sides that will satisfy the requirement. Since each of the four dice rolls are independent of one another, we can calculate the probability that none of them exceeds x to be the product $\left(\frac{x}{6}\right)\left(\frac{x}{6}\right)\left(\frac{x}{6}\right)\left(\frac{x}{6}\right) = \left(\frac{x}{6}\right)^4$, as desired

(ii)

x	1	2	3	4	5	6
P(X = x)	1/1296	15/1296	65/1296	175/1296	369/1296	671/1296

(iii)

$$\begin{aligned} E(X) &= \frac{1}{1296} + \frac{2 \times 15}{1296} + \frac{3 \times 65}{1296} + \frac{4 \times 175}{1296} + \frac{5 \times 369}{1296} + \frac{6 \times 671}{1296} \\ &= \frac{6797}{1296} \\ &\doteq 5.24 \end{aligned}$$

2.7 Binomial expansions

Recall: A binomial expression has **2 terms**.

Consider the expansion of binomial expressions of the form

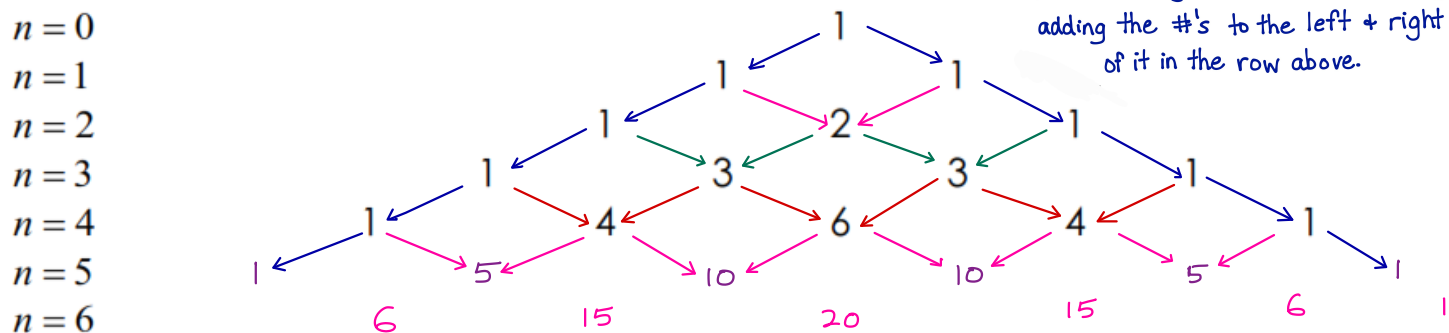
$n=0$	$(a + b)^0 = 1$	# terms 1 term
$n=1$	$(a + b)^1 = a + b$	2 terms
$n=2$	$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$	3 terms
$n=3$	$(a + b)^3 = (a + b)^2(a + b) = a^3 + 3a^2b + 3ab^2 + b^3$	4 terms
$n=4$	$(a + b)^4 = (a + b)^3(a + b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$	5 terms

and so on.

Note:

- 1) All expansions proceed in **decreasing powers of a and increasing powers of b**, their exponents always adding to the exponent on the binomial.
- 2) All the **coefficients** are precisely the **numbers in the Pascal's triangle** (see below).
- 3) The **number of terms in the expansion is $n+1$** .
- 4) In each term, the **sum of the exponents is n** .

The coefficients of the terms can be written as a triangular array of numbers known as Pascal's triangle.



A general expression for the **coefficient of the $(r+1)^{\text{th}}$ term** in the expansion of $(a+x)^n$ is:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad \text{where } n! = n(n-1)(n-2)\dots 3 \times 2 \times 1 \quad \text{(called } n \text{ factorial)}$$

and where, by definition, $0! = 1$.

Example: $5! = 5 \times 4 \times 3 \times 2 \times 1$

Note: $\binom{n}{r}$ read as "**n choose r**" appears on calculators as nC_r or nCr .

$$\begin{aligned} \binom{3}{2} &= \frac{3!}{2!(3-2)!} \\ &= \frac{3!}{2!1!} \\ &= \frac{3(2!)}{2!} \\ &= 3 \end{aligned}$$

General formula for the expansion of $(a+x)^n$ where $n \in \mathbb{Z}^+$:

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \dots + \boxed{\binom{n}{r}a^{n-r}x^r} + \dots + \binom{n}{n}x^n$$

t_1 t_2 t_3 t_4 t_{r+1} t_{n+1}

Note: There are $n+1$ terms. The above expansion is often known as the **binomial theorem**.

General formula for the expansion of $(1+x)^n$ where $n \in \mathbb{Z}^+$

$$(1+x)^n = \binom{n}{0}1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots + \binom{n}{n}x^n$$

The general term in the expansion of $(a+x)^n$ is $t_{r+1} = \binom{n}{r}a^{n-r}x^r$.

Example 1: Find the coefficients of the x^3 and x^5 terms in the expansion of $(1+x)^6$.

$$(1+x)^6 = \binom{6}{0}(1)^6(x)^0 + \binom{6}{1}(1)^5(x)^1 + \binom{6}{2}(1)^4x^2 + \binom{6}{3}(1)^3(x)^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6$$

$$= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

The coefficient of x^3 is 20 and that of x^5 is 6

Example 2: Write down the expansions of:

a) $(a+b)^7$

$$= \binom{7}{0}a^7 + \binom{7}{1}a^6b + \binom{7}{2}a^5b^2 + \binom{7}{3}a^4b^3 + \binom{7}{4}a^3b^4 + \binom{7}{5}a^2b^5 + \binom{7}{6}ab^6 + \binom{7}{7}b^7$$

$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

b) $(x-2)^6$

$$= [x+(-2)]^6$$

$$= \binom{6}{0}x^6 + \binom{6}{1}x^5(-2) + \binom{6}{2}x^4(-2)^2 + \binom{6}{3}x^3(-2)^3 + \binom{6}{4}x^2(-2)^4 + \binom{6}{5}x(-2)^5 + \binom{6}{6}x^0(-2)^6$$

$$= x^6 + 6x^5(-2) + 15x^4(4) + 20x^3(-8) + 15x^2(16) + 6x(-32) + 64$$

$$= x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x + 64$$

Example 3: Use the binomial theorem to find the first four terms in the expansion of $\left(2x - \frac{1}{x}\right)^9$.

$$= \binom{9}{0}(2x)^9 + \binom{9}{1}(2x)^8\left(-\frac{1}{x}\right) + \binom{9}{2}(2x)^7\left(-\frac{1}{x}\right)^2 + \binom{9}{3}(2x)^6\left(-\frac{1}{x}\right)^3 + \dots$$

$$= 512x^9 + 9(256x^8)\left(-\frac{1}{x}\right) + 36(128x^7)\left(-\frac{1}{x^2}\right) + 84(64x^6)\left(-\frac{1}{x^3}\right) + \dots$$

$$= 512x^9 - 2304x^7 + 4608x^5 - 5376x^3 + \dots$$

Example 4: Determine the coefficient of x^4 in the expansion of $(2-3x)^7$.

General term is $t_{r+1} = \binom{7}{r}(2)^{7-r}(-3x)^r$

Recall
 $t_{r+1} = \binom{n}{r}a^{n-r}x^r$

$$= \binom{7}{r}(2)^{7-r}(-3)^r(x)^r$$

coefficient

$\therefore (x)^r = x^4$
 $\therefore r = 4$
 $\therefore$ the coefficient of x^4 is: $\binom{7}{4}(2)^{7-4}(-3)^4$
 $= (35)(8)(81)$
 $= 22680$

Example 5: Calculate the value of the constant a if the coefficient of the x^3 term in the expansion $(a+2x)^4$ is 160.

$$\begin{aligned} \text{General term is } t_{r+1} &= \binom{4}{r} (a)^{4-r} (2x)^r & \because x^r = x^3 \\ &= \binom{4}{r} (a)^{4-r} (2)^r (x)^r & \because r=3 \\ & \quad \text{coefficient} & \therefore t_4 = \binom{4}{3} a^{4-3} (2)^3 x^3 \\ & & = 4a (8) x^3 \\ & & = 32 a x^3 \end{aligned}$$

$$\text{Equating coefficients: } 160 = 32a$$

$$a = \frac{160}{32}$$

$$= 5$$

$\therefore$ the value of a is 5.

Approximations: Use a binomial expansion to provide a good approximation to decimal quantities raised to a high power.

Example 6: Expand $(1+4x)^{14}$ in ascending powers of x , up to and including the 4th term. Hence, evaluate $(1.0004)^{14}$, correct to 4 decimal places.

$$\begin{aligned} (1+4x)^{14} &= 1 + \binom{14}{1} 4x + \binom{14}{2} (4x)^2 + \binom{14}{3} (4x)^3 + \dots \\ &= 1 + 14(4x) + 91(16x^2) + 364(64x^3) + \dots \\ &= 1 + 56x + 1456x^2 + 23296x^3 + \dots \end{aligned}$$

$$\therefore (1.0004)^{14} = [1 + 4(0.0001)]^{14} \quad x = 0.0001$$

$$\begin{aligned} \therefore (1.0004)^{14} &= 1 + 56(0.0001) + 1456(0.0001)^2 + 23296(0.0001)^3 + \dots \\ &= 1.005614583 \\ &\doteq 1.0056 \end{aligned}$$

Exit Card!

1. The first three terms in the expansion of $(1 - ax)^n$ are $t_1 = 1$, $t_2 = -12x$, and $t_3 = 63x^2$.

Use the general term to determine a and n .

$$(1 - ax)^n = \binom{n}{0}(-ax)^0 + \binom{n}{1}(-ax)^1 + \binom{n}{2}(-ax)^2 + \dots$$

$$= 1 - nax + \frac{n!}{(n-2)!2!} a^2 x^2 + \dots$$

$$t_1 = 1, \quad t_2 = -nax, \quad t_3 = \frac{n(n-1)}{2} a^2 x^2$$

Given $t_1 = 1, \quad t_2 = -12x, \quad t_3 = 63x^2$

$$-12x = -nax$$

$$na = 12 \quad \text{--- (1)}$$

$$a = \frac{12}{n} \quad \text{--- (2)}$$

$$63 = \frac{n(n-1)}{2} a^2 \quad \text{--- (3)}$$

Sub (2) in (3)

$$63 = \frac{n(n-1)}{2} \left(\frac{12}{n}\right)^2$$

$$63n = 72(n-1)$$

$$7n = 8n - 8$$

$$\boxed{n=8}$$

Sub in (2)

$$a = \frac{12}{8}$$

$$\boxed{a = \frac{3}{2}}$$

2. What is the numerical coefficient for the term containing x^2y^3 in the expansion of $(x - 3y)^5$?

$$t_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$t_{r+1} = \binom{5}{r} (x)^{5-r} (-3y)^r$$

$$= \binom{5}{r} (-3)^r x^{5-r} y^r$$

$$x^{5-r} y^r = x^2 y^3$$

$$\boxed{r=3}$$

$$\text{Coefficient} = \binom{5}{3} (-3)^3$$

$$= \frac{5!}{3!2!} (-27)$$

$$= \frac{5 \times 4}{2} \times (-27)$$

$$= -270$$

Coefficient of x^2y^3 term is -270

2.7 Practice

1. Expand and simplify the first three terms of $(x^2 - 3\sqrt{x})^{10}$. $[x^{20} - 30\sqrt{x^{37}} + 405x^{17} + \dots]$
2. In the expansion of $\left(x^2 + \frac{1}{\sqrt{x}}\right)^{10}$ find:
 - a) the constant term [45]
 - b) the middle term $\left[252x^{\frac{15}{2}}\right]$
3. In the expansion of $(1+3x)^n$, the coefficient of the term x^2 is $135n$, where $n \in \mathbb{Z}^+$. Find n . [31]
4. A game consists of cutting a shuffled deck of cards and receiving the dollar value of the face value of the card. Jacks are considered to be worth 11, queens 12, kings 13 and aces 1. What is the expectation of this game?[7]
5. In a manufacturing process, it is estimated that 0.1% of the products are defective. If a client places an order for 25 of the products, what is the probability that at least one of them is defective?
6. A game consists of rolling a pair of dice 10 times. For each sum that equals 6, 7 or 8 on the 2 dice, you win 1 dollar. If it costs 5 dollars to play the game, is this a fair game?
7. Consider the expansion of the expression $(x^3 - 3x)^6$.
 - a) Write down the number of terms in this expansion. [7]
 - b) Find the term in x^{12} . [-540]

Probability Review Problems

- For events A and B , the probabilities are $P(A) = \frac{3}{11}$, $P(B) = \frac{4}{11}$. Calculate the value of $P(A \cap B)$ if
 - $P(A \cup B) = \frac{6}{11}$;
 - events A and B are independent.
- Consider events A, B such that $P(A) \neq 0$, $P(A) \neq 1$, $P(B) \neq 0$, and $P(B) \neq 1$. In each of the situations (a), (b), (c) below state whether A and B are mutually exclusive (M); independent (I); neither (N).
 - $P(A|B) = P(A)$
 - $P(A \cap B) = 0$
 - $P(A \cap B) = P(A)$
- Let A and B be events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{3}{4}$ and $P(A \cup B) = \frac{7}{8}$.
 - Calculate $P(A \cap B)$.
 - Calculate $P(A|B)$.
 - Are the events A and B independent? Give a reason for your answer.
- Let A and B be independent events such that $P(A) = 0.3$ and $P(B) = 0.8$.
 - Find $P(A \cap B)$.
 - Find $P(A \cup B)$.
 - Are A and B mutually exclusive? Justify your answer.
- Two unbiased 6-sided dice are rolled, a red one and a black one. Let E and F be the events
 E : the same number appears on both dice;
 F : the sum of the numbers is 10.
 Find (a) $P(E)$; (b) $P(F)$; (c) $P(E \cup F)$.
- Two fair dice are thrown and the number showing on each is noted. The sum of these two numbers is S . Find the probability that
 - S is less than 8;
 - at least one die shows a 3;
 - at least one die shows a 3, given that S is less than 8.
- A painter has 12 tins of paint. Seven tins are red and five tins are yellow. Two tins are chosen at random. Calculate the probability that both tins are the same color.
- A class contains 13 girls and 11 boys. The teacher randomly selects four students. Determine the probability that all four students selected are girls.
- In a survey of 200 people, 90 of whom were female, it was found that 60 people were unemployed, including 20 males.
 - Using this information, complete the table below.

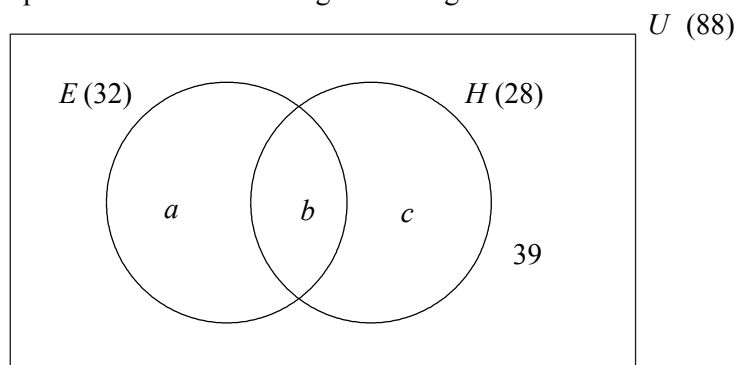
	Males	Females	Totals
Unemployed			
Employed			
Totals			200

- If a person is selected at random from this group of 200, find the probability that this person is
 - an unemployed female;
 - a male, given that the person is employed.

10. A discrete random variable X has a probability distribution as shown in the table below.

x	0	1	2	3
$P(X=x)$	0.1	a	0.3	b

- (a) Find the value of $a + b$.
 (b) Given that $E(X) = 1.5$, find the value of a and of b .
11. Two ordinary, 6-sided dice are rolled and the total score is noted. Find the probability of getting one or more sixes. Draw a tree diagram to support your answer.
12. A packet of seeds contains 40% red seeds and 60% yellow seeds. The probability that a red seed grows is 0.9, and that a yellow seed grows is 0.8. A seed is chosen at random from the packet.
 (a) Calculate the probability that the chosen seed is red and grows.
 (b) Calculate the probability that the chosen seed grows.
 (c) Given that the seed grows, calculate the probability that it is red.
13. A bag contains four apples (A) and six bananas (B). A fruit is taken from the bag and eaten. Then a second fruit is taken and eaten. Find the probability that one of each type of fruit was eaten.
14. In a class, 40 students take chemistry only, 30 take physics only, 20 take both chemistry and physics, and 60 take neither.
 (a) Find the probability that a student takes physics given that the student takes chemistry.
 (b) Find the probability that a student takes physics given that the student does **not** take chemistry.
 (c) State whether the events “taking chemistry” and “taking physics” are mutually exclusive, independent, or neither. Justify your answer.
15. The heights, H , of the people in a certain town are normally distributed with mean 170 cm and standard deviation 20 cm.
 (a) A person is selected at random. Find the probability that his height is less than 185 cm.
 (b) Given that $P(H > d) = 0.6808$, find the value of d .
16. If $n(U) = 33$, $n(A \cup B) = 29$, $n(A \cap B) = 5$, and $n(B^c) = 23$, Find $n(A^c)$.
17. In a school of 88 boys, 32 study economics (E), 28 study history (H) and 39 do not study either subject. This information is represented in the following Venn diagram.



- (a) Calculate the values a , b , c .
 (b) A student is selected at random.
 (i) Calculate the probability that he studies **both** economics and history.
 (ii) Given that he studies economics, calculate the probability that he does **not** study history.
 (c) A group of three students is selected at random from the school.
 (i) Calculate the probability that none of these students studies economics.
 (ii) Calculate the probability that at least one of these students studies economics.

- 18.** A game is played, where a die is tossed and a marble selected from a bag.
 Bag M contains 3 red marbles (R) and 2 green marbles (G).
 Bag N contains 2 red marbles and 8 green marbles.
 A fair six-sided die is tossed. If a 3 or 5 appears on the die, bag M is selected (M).
 If any other number appears, bag N is selected (N).
 A single marble is then drawn at random from the selected bag.
- Write down the probability that bag M is selected and a green marble drawn from it.
 - Find the probability that a green marble is drawn from either bag.
 - Given that the marble is green, calculate the probability that it came from Bag M.
 - A player wins \$2 for a red marble and \$5 for a green marble. What are his expected winnings?
- 19.** A fair coin is tossed 10 times. Find the probability of getting
- Exactly 6 heads
 - at least 2 heads
 - at most 8 heads
- 20.** The probability distribution of a discrete random variable X is given by $P(X = x) = \frac{x^2}{14}$ $x \in \{1, 2, k\}$,
 where $k > 0$. Find the value of k .
- 21.** The heights of trees in a forest are normally distributed with mean height 17 metres. One tree is selected at random. The probability that a selected tree has a height greater than 24 metres is 0.06.
- Find the probability that the tree selected has a height less than 24 metres.
 - The probability that the tree has a height less than D metres is 0.06. Find the value of D
- 22.** A random variable X is distributed normally with a mean of 100 and a variance of 100.
- Find the value of X that is 1.12 standard deviations above the mean.
 - Find the value of X that is 1.12 standard deviations below the mean.
- 23.** Let A and B be independent events, where $P(A) = 0.6$ and $P(B) = x$.
- Write down an expression for $P(A \cap B)$.
 - Given that $P(A \cup B) = 0.8$,
 - find x ;
 - find $P(A \cap B)$.
 - Hence, explain why A and B are not mutually exclusive
- 24.** Let X be normally distributed with mean 100 cm and standard deviation 5 cm.
- On the diagram below, shade the region representing $P(X > 105)$.



(b) Given that $P(X < d) = P(X > 105)$, find the value of d .

(c) Given that $P(X > 105) = 0.16$ (correct to two significant figures), find $P(d < X < 105)$.

25. In any given season, a soccer team plays 65 % of their games at home. When the team plays at home, they win 83 % of their games. When they play away from home, they win 26 % of their games. The team plays one game.

(a) Find the probability that the team wins the game.

(b) If the team does not win the game, find the probability that the game was played at home.

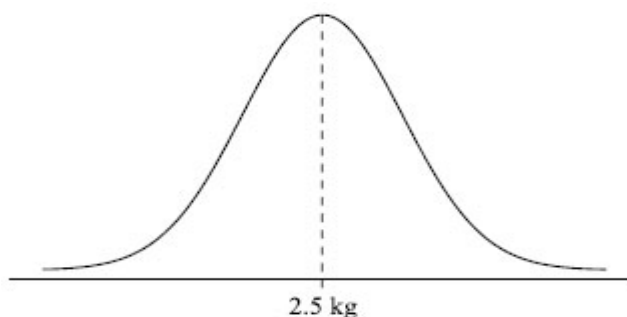
26. The weights of chickens for sale in a shop are normally distributed with mean 2.5 kg and standard deviation 0.3 kg.

(a) A chicken is chosen at random.

(i) Find the probability that it weighs less than 2 kg.

(ii) Find the probability that it weighs more than 2.8 kg.

(iii) Shade the areas that represent the probabilities from parts (i) and (ii).



(iv) **Hence** show that the probability that it weighs between 2 kg and 2.8 kg is 0.7936 (to four significant figures).

(b) A customer buys 10 chickens.

(i) Find the probability that all 10 chickens weigh between 2 kg and 2.8 kg.

(ii) Find the probability that at least 7 of the chickens weigh between 2 kg and 2.8 kg.

27. Find the value of k and n if $(1 + kx)^n = 1 - 4x + 7x^2 + \dots$

28. Find the coefficient of x^2 in $(2 + x)^4 \left(1 + \frac{1}{x^2}\right)$.

29. Find the coefficient of x^4 in $\left(3x^2 - \frac{2}{x}\right)^5$.

30. Consider the expansion of the expression $(x^3 - 3x)^6$

(a) Write down the number of terms in this expansion.

(b) Find the term in x^{12}

Probability Review Problems -Solution

$$\begin{aligned}
 1. \quad (a) \quad & P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B) \\
 &= \frac{3}{11} + \frac{4}{11} - \frac{6}{11} \\
 &= \frac{1}{11} \quad (0.0909)
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad & \text{For independent events, } P(A \cap B) = P(A) \times P(B) \\
 &= \frac{3}{11} \times \frac{4}{11} \\
 &= \frac{12}{121} \quad (0.0992)
 \end{aligned}$$

2. (a) Independent (I)
 (b) Mutually exclusive (M)
 (c) Neither (N)

$$\begin{aligned}
 3. \quad (a) \quad & P(A \cup B) = P(A) + P(B) - P(A \cap B) \\
 & P(A \cap B) = \frac{1}{2} + \frac{3}{4} - \frac{7}{8} = \frac{3}{8}
 \end{aligned}$$

$$(b) \quad P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{3}{8}}{\frac{3}{4}} = \frac{1}{2}$$

- (c) Yes, the events are independent

$$\text{EITHER } P(A|B) = P(A) \quad \text{OR} \quad P(A \cap B) = P(A)P(B)$$

$$\begin{aligned}
 4. \quad (a) \quad & \text{Independent} \Rightarrow P(A \cap B) = P(A) \times P(B) \quad (= 0.3 \times 0.8) \\
 &= 0.24
 \end{aligned}$$

$$\begin{aligned}
 (b) \quad & P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (= 0.3 + 0.8 - 0.24) \\
 &= 0.86
 \end{aligned}$$

- (c) No, **with** valid reason eg $P(A \cap B) \neq 0$ or $P(A \cup B) \neq P(A) + P(B)$ or correct numerical equivalent

5. Total number of possible outcomes = 36 (may be seen anywhere)

$$(a) \quad P(E) = P(1, 1) + P(2, 2) + P(3, 3) + P(4, 4) + P(5, 5) + P(6, 6) = \frac{6}{36}$$

$$(b) \quad P(F) = P(6, 4) + P(5, 5) + P(4, 6) = \frac{3}{36}$$

$$(c) \quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$P(E \cap F) = \frac{1}{36}$$

$$P(E \cup F) = \frac{6}{36} + \frac{3}{36} - \frac{1}{36} \left(= \frac{8}{36} = \frac{2}{9}, 0.222 \right)$$

6. Sample space = $\{(1, 1), (1, 2) \dots (6, 5), (6, 6)\}$
 (This may be indicated in other ways, for example, a grid or a tree diagram, partly or fully completed)

$$(a) \quad P(S < 8) = \frac{6 + 5 + 4 + 3 + 2 + 1}{36} = \frac{7}{12}$$

OR

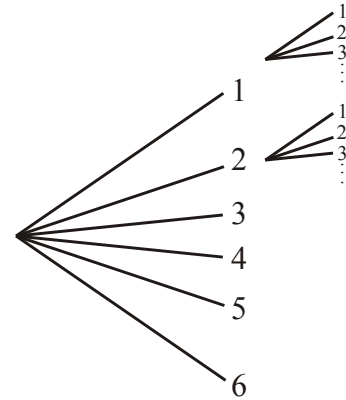
$$P(S < 8) = \frac{7}{12}$$

$$(b) \quad P(\text{at least one } 3) = \frac{1 + 1 + 6 + 1 + 1 + 1}{36} = \frac{11}{36}$$

OR

$$P(\text{at least one } 3) = \frac{11}{36}$$

$$(c) \quad P(\text{at least one } 3 \mid S < 8) = \frac{P(\text{at least one } 3 \cap S < 8)}{P(S < 8)} = \frac{\frac{7}{36}}{\frac{7}{12}} = \frac{1}{3}$$



$$7. \quad P(RR) = \frac{7}{12} \times \frac{6}{11} \left(= \frac{7}{22} \right)$$

$$P(YY) = \frac{5}{12} \times \frac{4}{11} \left(= \frac{5}{33} \right)$$

$$P(\text{same colour}) = P(RR) + P(YY) \\ = \frac{31}{66} (= 0.470, 3 \text{ sf})$$

8. Correct probabilities $\left(\frac{13}{24}\right), \left(\frac{12}{23}\right), \left(\frac{11}{22}\right), \left(\frac{10}{21}\right)$

$$\text{Multiplying } \left(\frac{13}{24} \times \frac{12}{23} \times \frac{11}{22} \times \frac{10}{21}\right)$$

$$P(4 \text{ girls}) = \frac{17160}{255024} \left(= \frac{65}{966} = 0.0673 \right)$$

9. (a)

	Males	Females	Totals
Unemployed	20	40	60
Employed	90	50	140
Totals	110	90	200

$$(b) \quad (i) \quad P(\text{unemployed female}) = \frac{40}{200} = \frac{1}{5}$$

$$(ii) \quad P(\text{male I employed person}) = \frac{90}{140} = \frac{9}{14}$$

10. (a) For summing to 1

$$\text{eg } 0.1 + a + 0.3 + b = 1$$

$$a + b = 0.6$$

(b) $E(X) = \sum x f(x)$

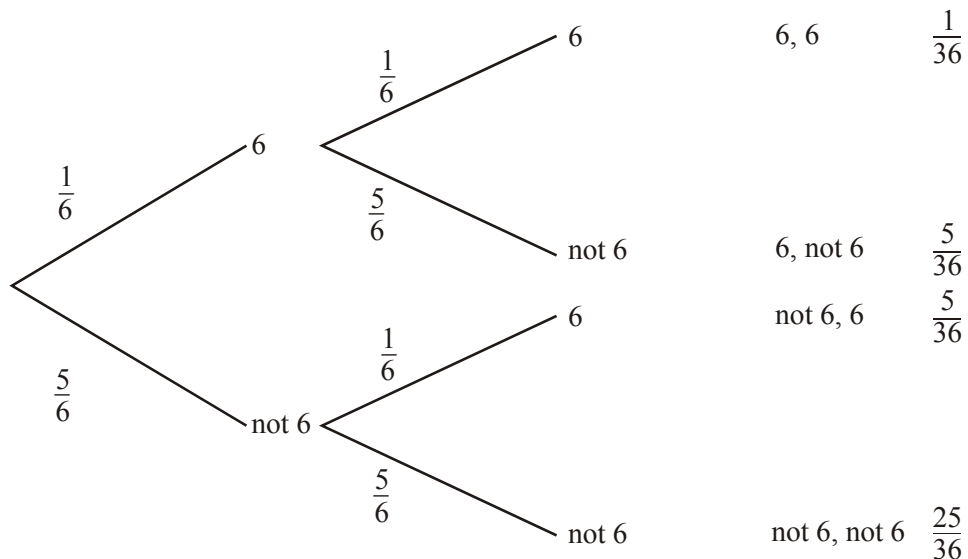
$$0 \times 0.1 + 1 \times a + 2 \times 0.3 + 3 \times b, 0.1 + a + 0.6 + 3b = 1.5$$

$$0 + a + 0.6 + 3b = 1.5 \quad (a + 3b = 0.9)$$

Solving simultaneously gives

$$a = 0.45 \quad b = 0.15$$

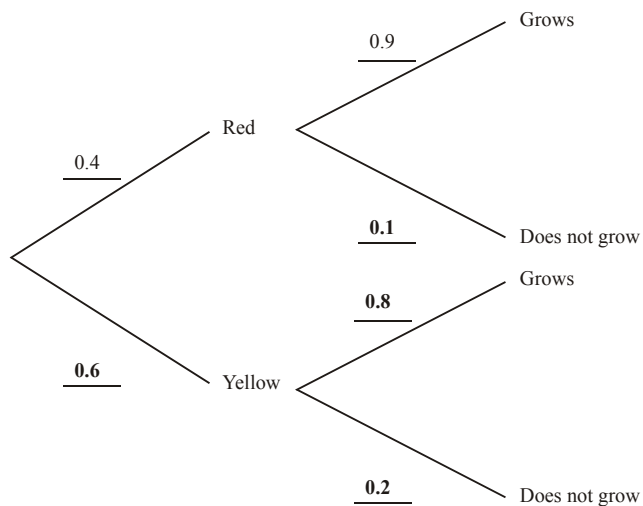
11. (a)



(b) $P(\text{one or more sixes}) = \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \quad \text{or} \quad \left(1 - \frac{5}{6} \times \frac{5}{6}\right)$

$$= \frac{11}{36}$$

12. (a)



(b) (i) 0.4×0.9

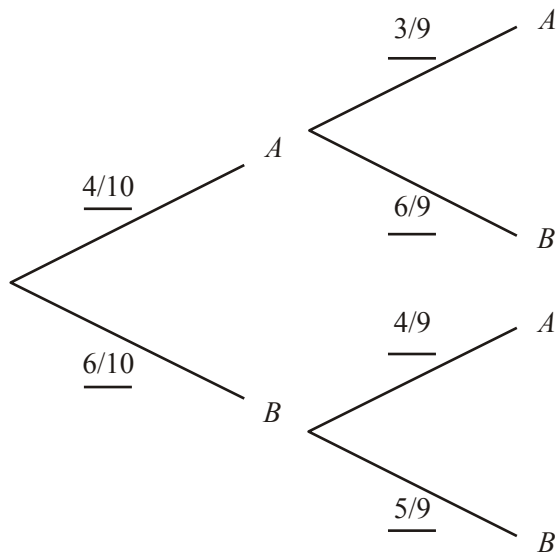
$$= 0.36 \text{ (A1)}$$

(ii) $0.36 + 0.6 \times 0.8 \quad (= 0.36 + 0.48)$

$$= 0.84 \text{ (A1)}$$

$$(iii) \frac{P(\text{red} \cap \text{grows})}{P(\text{grows})} = \frac{0.36}{0.84} = 0.429 \left(\frac{3}{7} \right)$$

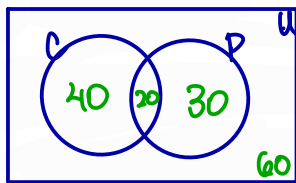
13. (a)



$$(b) \left(\frac{4}{10} \times \frac{6}{9} \right) + \left(\frac{6}{10} \times \frac{4}{9} \right)$$

$$= \frac{48}{90} \left(\frac{8}{15}, 0.533 \right)$$

14.



$n(U) = 150$

Let P be the event the student takes Physics.
Let C be the event the student takes Chemistry.

$$a) P(P|C) = \frac{P(P \cap C)}{P(C)}$$

$$= \frac{20}{60}$$

$$= \frac{1}{3}$$

$$b) P(P|C) = \frac{P(P \cap C)}{P(C)}$$

$$= \frac{30}{90}$$

$$= \frac{1}{3}$$

15. (a) $z = \frac{185-170}{20} = 0.75$

$$P(Z < 0.75) = 0.773$$

(b) $z = -0.47$

$$-0.47 = \frac{d-170}{20}$$

$$d = 161$$

$$c) P(P \cup C) = \frac{30}{150} + \frac{40}{150} - \frac{20}{150}$$

$$= \frac{1}{3}$$

$$P(P) + P(C) = \frac{50}{150} + \frac{60}{150}$$

$$= \frac{11}{15}$$

$$\therefore P(P \cup C) \neq P(P) + P(C)$$

$\therefore$ Not mutually exclusive

(OR explain that $P \cap C \neq 0$
 $\therefore$ not mutually exclusive)

$$P(P \cap C) = \frac{2}{15}$$

$$P(P) \cdot P(C) = \frac{50}{150} \times \frac{60}{150}$$

$$= \frac{2}{15}$$

$$\therefore P(P \cap C) = P(P) \cdot P(C)$$

$\therefore$ The two events are independent.

$$16. \quad n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$29 = n(A) + (33 - 23) - 5$$

$$n(A) = 24$$

$$n(A^c) = n(U) - n(A)$$

$$= 33 - 24$$

$$= 9$$

$$17. a) \quad n(E \cup H) = a + b + c = 88 - 39 = 49$$

$$n(E \cup H) = 32 + 28 - b = 49$$

$$60 - 49 = b = 11$$

$$a = 32 - 11 = 21$$

$$c = 28 - 11 = 17$$

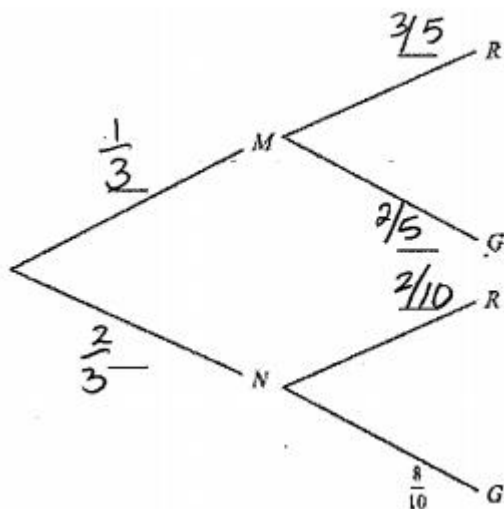
$$b)) (i) \quad P(E \cap H) = \frac{11}{88} = \frac{1}{8}$$

$$(ii) \quad P(H' | E) = \frac{P(H' \cap E)}{P(E)} = \frac{\frac{21}{88}}{\frac{32}{88}} = \frac{21}{32} (= 0.656)$$

$$c) \quad (i) \quad P(\text{none in economics}) = \frac{56 \times 55 \times 54}{88 \times 87 \times 86} = 0.253$$

$$(ii) \quad P(\text{at least one}) = 1 - 0.253 = 0.747$$

18.



$$a) (i) \quad \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$$

$$(ii) \quad \frac{1}{3} \left(\frac{2}{5} \right) + \frac{2}{3} \left(\frac{8}{10} \right) = \frac{2}{3}$$

$$(iii) \quad p(M | G) = \frac{P(M \cap G)}{P(G)} = \frac{\frac{2}{15}}{\frac{2}{3}} = \frac{1}{5}$$

$$b) \quad P(R) = \frac{1}{3} \left(\frac{3}{5} \right) + \frac{2}{3} \left(\frac{2}{10} \right) = \frac{1}{3}$$

$$P(G) = \frac{2}{3}$$

$$E(\text{Red or Green}) = \$2 \left(\frac{1}{3} \right) + \$5 \left(\frac{2}{3} \right) = \$4$$

19. If we assume the coin tosses are independent, then the number of heads in ten flips is a random variable $X \sim B(10, 1/2)$ and $P(X = k) = \binom{10}{k} 2^{-10}$.

a) $P(X = 6) = \binom{10}{6} 2^{-10} = \frac{210}{1024} \doteq 0.205$

b)
$$P(X \geq 2) = 1 - \binom{10}{1} 2^{-10} - \binom{10}{0} 2^{-10}$$
$$= \frac{1013}{1024} \doteq 0.989$$

c) $P(X \leq 8) = P(X \geq 2) = 0.989$

20. $k=3$

21. a) $P(X < 24) = 1 - 0.06 = 0.94$

b) $D=10$

(Q2) $X \sim N(100, 10^2)$

(A) $x = 100 + 1.12(10) = 111.2$ (B) $x = 100 - 1.12(10) = 88.8$

(Q3) $P(A) = 0.6$ $P(B) = x$
A and B are Indep.

$P(A \cap B) = P(A) \cdot P(B|A)$ If A is first

$P(A \cap B) = P(B) \cdot P(A|B)$ If B is first

In either case since A and B are Indep, given info is redundant.

$\therefore P(A \cap B) = P(A) \cdot P(B)$ regardless of event order

$\therefore P(A \cap B) = 0.6x$

If $P(A \cup B) = 0.8$ then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$0.8 = 0.6 + x - 0.6x$

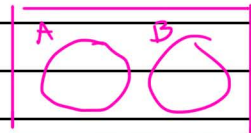
$0.2 = 0.4x$

$x = 1/2$

$\therefore P(A \cap B) = 0.6(1/2)$
 $= 0.3$

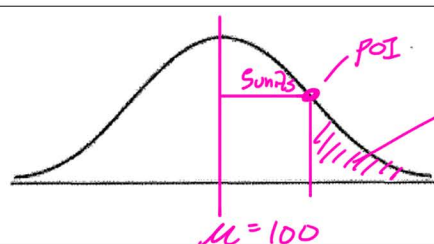
A and B are mutually exclusive if $P(A \cap B) = \emptyset$

Since $P(A \cap B) \neq \emptyset$, they cannot be mutually exclusive.



(A)

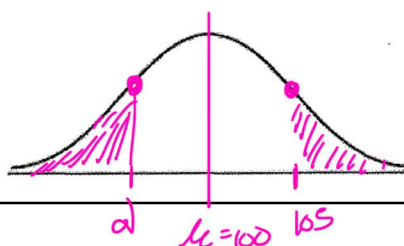
(24) $X \sim N(100, 5^2)$



Area = $P(X > 105)$

(B) If $P(X < d) = P(X > 105)$ then due to symmetry,

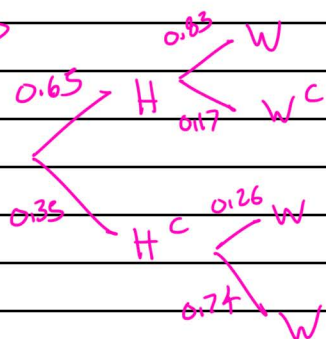
$$d = 100 - 5 = 95$$



(C) $P(X > 105) = 0.16$

$$\begin{aligned} P(d < X < 105) &= P(X < 105) - P(X < d) \\ &= 1 - P(X \geq 105) - P(X > 105) \\ &= 1 - 2(0.16) \\ &= 0.68 \end{aligned}$$

(25) *



$$\begin{aligned} P(W) &= P(H)P(W|H) + P(H^c)P(W|H^c) \\ &= (0.65)(0.83) + (0.35)(0.26) \\ &= 0.6305 \\ &= 63.05\% \end{aligned}$$

sorry for bad
Math
Form

(B) $P(H|W^c) = \frac{P(H \cap W^c)}{P(W^c)} = \frac{(0.65)(0.17)}{1 - 0.6305} = \frac{0.221}{0.3695} \approx 0.2991$
or 29.91%

(26) let the random variable X rep the weight of the chicken
 $X \sim N(2.5, 0.3^2)$

$$P(X < 2) = \text{NormalCDF}(-1E99, 2, 2.5, 0.3) \\ = 0.0477903304$$

$$P(X > 2.8) = \text{NormalCDF}(2.8, 1E99, 2.5, 0.3) \\ = 0.1586552596$$

(iii) Shade the areas that represent the probabilities from parts (i) and (ii).



$$P(2 < X < 2.8) = 1 - P(X < 2) - P(X > 2.8) \\ = 1 - 0.04779 - 0.15865 \\ = 0.7936$$

(iv) Hence show that the probability that it weighs between 2 kg and 2.8 kg is 0.7936 (to four significant figures).

(27) let the random variable Y represent the # of chickens that weigh between 2 and 2.8 kg
 $Y \sim B(10, 0.7936)$

$$\text{i) } P(X=10) = \binom{10}{10} (0.7936)^{10} (1-0.7936)^0 \\ = \text{Binomial PDF}(10, 0.7936, 10) \\ = 0.0990869799$$

$$\text{ii) } P(X \geq 7) = 1 - P(X < 7) \\ = 1 - P(X \leq 6) \\ = 1 - \text{BinomialCDF}(10, 0.7936, 6) \\ = 0.8675835023$$

$$(27) (1+kx)^n = 1 - 4x + 7x^2 - \dots$$

$$(1+kx)^n = \sum_{i=0}^{\infty} \binom{n}{i} (1)^{n-i} (kx)^i$$

$$\therefore n(kx) = -4x$$

$$nk = -4$$

$$\binom{n}{2} (kx)^2 = 7x^2$$

$$\frac{n!}{2!(n-2)!} \cdot k^2 x^2 = 7x^2$$

$$\frac{n(n-1)}{2} \cdot k^2 = 7$$

$$(nk)(n-1)(k) = 14$$

$$(-4)(n-1)(k) = 14$$

$$(-4)(n-1)\left(-\frac{4}{n}\right) = 14$$

$$\frac{n-1}{n} = \frac{14}{16}$$

$$1 - \frac{1}{n} = \frac{7}{8}$$

$$\frac{1}{8} = \frac{1}{n}$$

$$\boxed{n=8}$$

$$\boxed{k = -\frac{1}{2}}$$

(28) $(2+x)^4 \left(1 + \frac{1}{x^2}\right)$ Find coeff of x^2

$$\left(1 + \frac{1}{x^2}\right) \left[\binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(x) + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)(x)^3 + \binom{4}{4}(x)^4 \right]$$

$$\left(1 + \frac{1}{x^2}\right) (16 + 32x + 24x^2 + 8x^3 + x^4)$$

x^2 term is created by $(1)(24x^2) + \left(\frac{1}{x^2}\right)(x^4)$

∴ The coefficient is 25

(29) Find the coefficient of x^4 in $\left(3x^2 - \frac{2}{x}\right)^5$

$$(3x^2 - 2x^{-1})^5 = \sum_{i=0}^5 \binom{5}{i} (3x^2)^{5-i} (-2x^{-1})^i$$

$$= \sum_{i=0}^5 \binom{5}{i} (3)^{5-i} (-2)^i (x)^{10-2i-i}$$

$$10-3i=4$$

$$6=3i$$

$$i=2$$

$$\binom{5}{2} (3)^3 (-2)^2 = 1080$$

∴ Coefficient of x^4 term is

(30) $(x^3 - 3x)^6 = \sum_{i=0}^6 \binom{6}{i} (x^3)^{6-i} (-3x)^i$

$$= \sum_{i=0}^6 \binom{6}{i} (-3)^i (x)^{18-3i+i}$$

$$= \sum_{i=0}^6 \binom{6}{i} (-3)^i (x)^{18-2i}$$

There are 7 terms in the binomial expansion

For x^{12} term, $i=3$

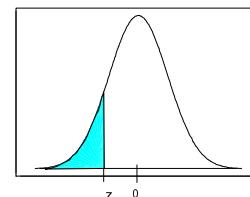
$$\binom{6}{3} (-3)^3 = -540$$

∴ The coefficient of the x^{12} term is -540

Appendix A

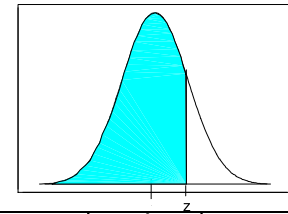
Standard Normal Cumulative Probability Table

Cumulative probabilities for **NEGATIVE** z-values are shown in the following table:



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Cumulative probabilities for POSITIVE z-values are in the following table:



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

Appendix B

Distribution Functions

A random variable Z has the standard normal distribution if it has the probability density function φ given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

The standard normal probability density function has the famous bell shape that is known to just about everyone.

The standard normal density function φ satisfies the following properties:

- φ is symmetric about $z=0$.
- φ increases and then decreases, with mode $z=0$.
- φ is concave upward and then downward and then upward again, with inflection points at $z=\pm 1$.
- $\varphi(z) \rightarrow 0$ as $z \rightarrow \infty$ and as $z \rightarrow -\infty$.

The standard normal distribution function Φ , given by

$$\Phi(z) = \int_{-\infty}^z \varphi(t) dt = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

The mean and variance of the standard normal distribution are¹

- $E(Z)=0$
- $\text{var}(Z)=1$

The General Normal Distribution

Suppose that $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ and that Z has the standard normal distribution. Then $X = \mu + \sigma Z$ has the **normal distribution** with mean μ and standard deviation σ .

For a particular value x of X , the distance from x to the mean μ of X expressed in units of standard deviation σ is.

- Of course, by symmetry, if Z has a mean, the mean must be 0, but we have to argue that the mean exists. Actually it's not hard to compute the mean directly. Note that

$$E(Z) = \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^0 z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \int_0^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

The integrals on the right can be evaluated explicitly using the simple substitution $u = z^2/2$. The result is

$$E(Z) = \frac{-1}{\sqrt{2\pi}} + \frac{-1}{\sqrt{2\pi}} = 0$$

- Note that $\text{var}(Z) = E(Z^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz$

Integrate by parts, using the parts $u=z$ and $dv=z\phi(z)dz$. Thus $du=dz$ and $v=-\phi(z)$. Note that $z\phi(z)\rightarrow 0$ as $z\rightarrow\infty$ and as $z\rightarrow-\infty$. Thus, the integration by parts formula gives

$$\text{var}(Z) = \int_{-\infty}^{\infty} \phi(z) dz = 1$$

Since we have subtracted off the mean (the center of the distribution) and factored out the standard deviation (the horizontal spread), this new value z is not only a rescaled version of x , but is also a realization of a standard normal random variable Z .

In this way, we can standardize any value from a generic normal distribution, transforming it into one from a standard normal distribution. Thus we reduce the problem of calculating probabilities for an event from a normal random variable to calculating probabilities for an event from a *standard* normal random variable.

The normal distribution has probability density function f given by

$$f(x) = \frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } x \in \mathbb{R}$$