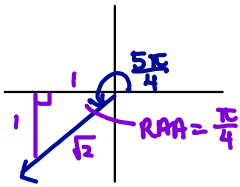


Trigonometry Warm Up

1. Determine the exact value of $\sin \frac{5\pi}{4}$.



$$\begin{aligned}\sin \frac{5\pi}{4} &= -\sin \frac{\pi}{4} \\ &= -\frac{1}{\sqrt{2}} \text{ or } -\frac{\sqrt{2}}{2}\end{aligned}$$

2. Solve: $\cos \theta = \frac{1}{2}$, $0 \leq \theta \leq 2\pi$

$$RAA = \frac{\pi}{3}$$

$\because \cos$ is + in Q1 and Q4

$$\therefore \theta_1 = \frac{\pi}{3}, \theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

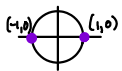
3. Solve: $2 \sin^2 \theta - \sin \theta = 0$, $0 \leq \theta \leq 3\pi$

$$\sin \theta (2 \sin \theta - 1) = 0$$

$$\therefore \sin \theta = 0$$

$$\text{or } 2 \sin \theta - 1 = 0$$

$$\sin \theta = \frac{1}{2}$$



$$\theta_1 = 0$$

$$\theta_2 = \pi$$

$$\theta_3 = 2\pi$$

$$\theta_4 = 3\pi$$

$$RAA = \frac{\pi}{6}$$

$\because \sin$ is + in Q1 & Q2

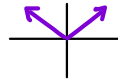
$$\therefore \theta_5 = \frac{\pi}{6}$$

$$\theta_6 = \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

$$\theta_7 = \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$$

$$\theta_8 = \frac{5\pi}{6} + 2\pi = \frac{17\pi}{6}$$



4. Prove: $\cos^4 x - \sin^4 x = 1 - 2 \sin^2 x$

$$LS = \cos^4 x - \sin^4 x$$

$$= (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x)$$

$$= (1)(1 - \sin^2 x - \sin^2 x)$$

$$= 1 - 2 \sin^2 x$$

$$= RS$$

$$\therefore LS = RS$$

$\therefore$ The identity is true

5. A Ferris Wheel has a radius of 10 m and completes one full revolution in 36 s. The riders board the ride from a platform 1 m above the ground at the bottom of the wheel.

- a) Determine the equation of the sine function that models the position of a rider above the ground, $h(t)$, in metres, at time t , in seconds.

$$|a| = 10$$

$$vd = +11$$

$$per = 36 \text{ s}$$

$$p.s. = \frac{1}{4}(36) = 9 \text{ s}$$

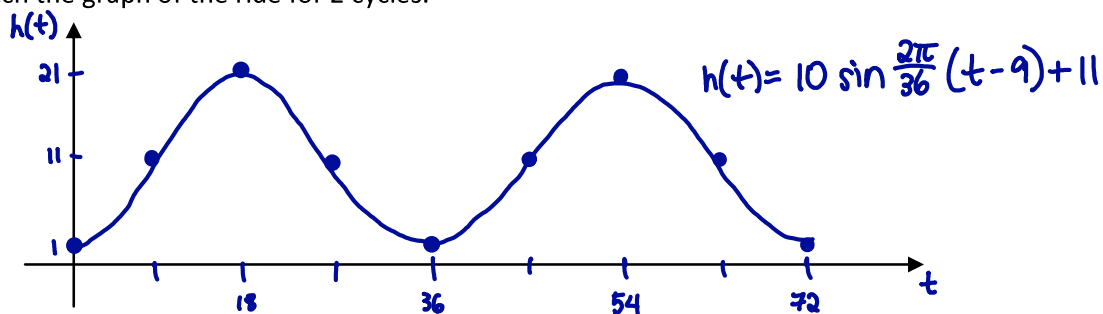
$$h(t) = 10 \sin \frac{2\pi}{36}(t-9) + 11 \quad \leftarrow \text{radians}$$

$$\text{or } h(t) = 10 \sin 10^\circ(t-9) + 11 \quad \leftarrow \text{degrees}$$

- b) When will the rider be at maximum height? 18 s

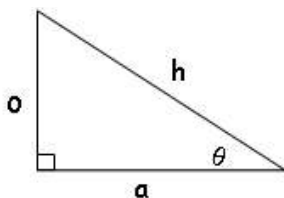
- c) What is the maximum height? 21 m

- d) Sketch the graph of the ride for 2 cycles.



5.1: Part I-Trigonometry –Double Angle Formulas

Primary Trigonometry Ratio (valid for right angle triangles only)



SINE

$$\sin(\theta) = \frac{o}{h}$$

COSINE

$$\cos(\theta) = \frac{a}{h}$$

TANGENT

$$\tan(\theta) = \frac{o}{a}$$

Reciprocal Trigonometric Ratios

The reciprocals of the primary trigonometric ratios sine, cosine and tangent are cosecant, secant and cotangent, respectively.

COSECANT

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

SECANT

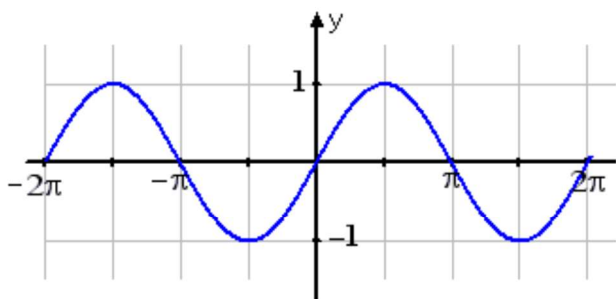
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

COTANGENT

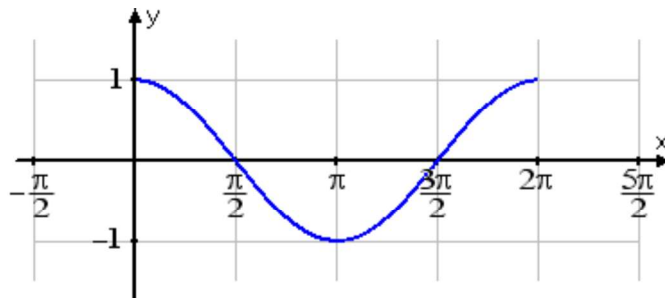
$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Graphs

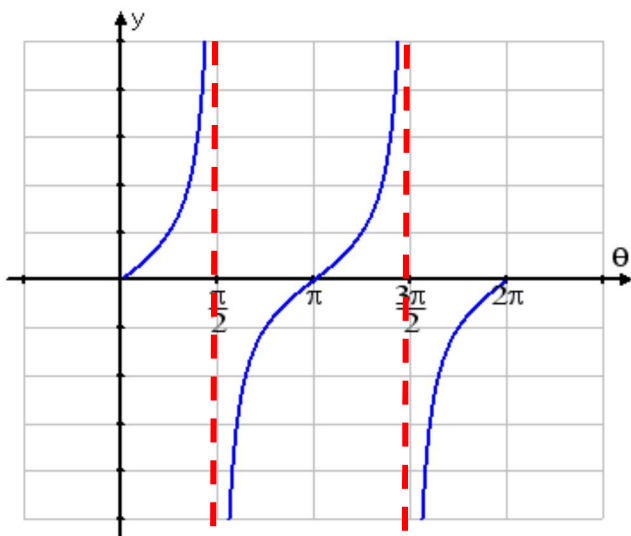
$y = \sin(\theta)$, Period = 2π



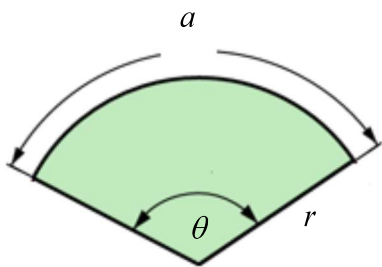
$y = \cos(\theta)$, Period = 2π



$y = \tan(\theta)$, Period = π

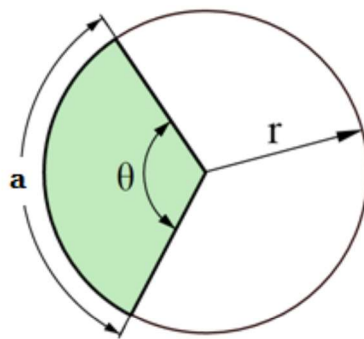


Area of Sector

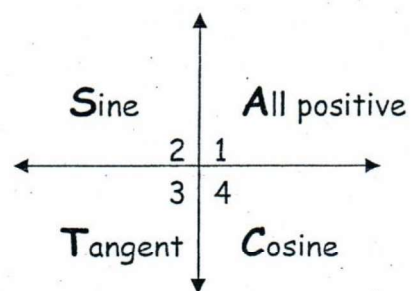
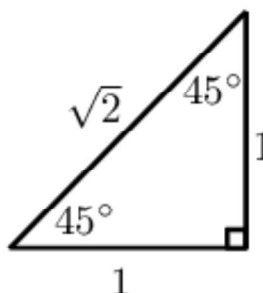
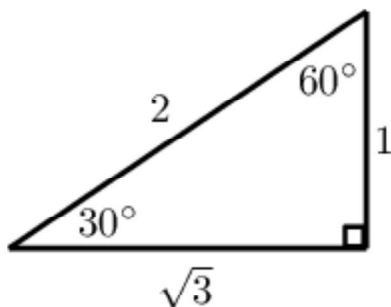


$$A = \frac{1}{2} r^2 \theta$$

$$\theta = \frac{a}{r}$$



Special Angles



Reciprocal Identities

$$\csc A = \frac{1}{\sin A}$$

$$\sec(A) = \frac{1}{\cos(A)}$$

$$\cot(A) = \frac{1}{\tan(A)}$$

$$\tan(A) = \frac{1}{\cot(A)}$$

$$\cot(A) = \frac{1}{\tan(A)}$$

Quotient Identities

$$\tan(A) = \frac{\sin(A)}{\cos(A)}$$

$$\cot(A) = \frac{\cos(A)}{\sin(A)}$$

Pythagorean Identity

$$\sin^2(A) + \cos^2(A) = 1$$

$$\tan^2(A) + 1 = \sec^2(A)$$

$$\cot^2(A) + 1 = \csc^2(A)$$

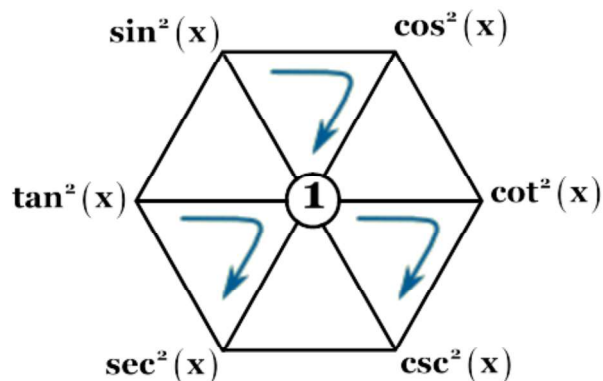
Reflection Identities

$$\sin(-A) = -\sin(A)$$

$$\cos(-A) = \cos(A)$$

$$\tan(-A) = -\tan(A)$$

The Hexagon Trig Trick



COMPOUND ANGLE IDENTITIES

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$$

DOUBLE ANGLE IDENTITIES

$$\sin(2A) = 2\sin(A)\cos(A)$$

$$\cos(2A) = \cos^2(A) - \sin^2(A)$$

$$\cos(2A) = 2\cos^2(A) - 1$$

$$\cos(2A) = 1 - 2\sin^2(A)$$

$$\tan(2A) = \frac{2\tan(A)}{1 - \tan^2(A)}$$

Example 1: Express the following as a single trigonometric ratio.

a) $2\sin(5A)\cos(5A)$
 $= \sin(10A)$

b) $1 - 2\sin^2(3x)$
 $= \cos(6x)$

c) $2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$
 $= \sin 2\left(\frac{x}{2}\right)$
 $= \sin(x)$

d) $\sin(2x)\cos(2x)$
 $= \frac{1}{2}\sin(4x)$

e) $2\cos^2(3\theta - 2) - 1$
 $= \cos 2(3\theta - 2)$
 $= \cos(6\theta - 4)$

f) $\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)$
 $= \cos 2\left(\frac{\theta}{2}\right)$
 $= \cos(\theta)$

Example 2: Express the following as a single trigonometric ratio and then evaluate.

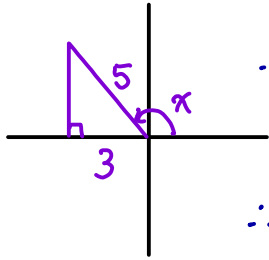
a) $\cos^2\left(\frac{\pi}{12}\right) - \sin^2\left(\frac{\pi}{12}\right)$
 $= \cos 2\left(\frac{\pi}{12}\right)$
 $= \cos\left(\frac{\pi}{6}\right)$
 $= \frac{\sqrt{3}}{2}$

b) $\frac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$
 $= \tan 2\left(\frac{\pi}{8}\right)$
 $= \tan\left(\frac{\pi}{4}\right)$
 $= 1$

Example 3: If $\cos(x) = \frac{3}{5}$, find the exact value of $\cos(4x)$.

$$\begin{aligned}\cos(4x) &= \cos[2(2x)] \\ &= 2\cos^2(2x) - 1 \\ &= 2[2\cos^2(x) - 1]^2 - 1 \\ &= 2\left[2\left(\frac{3}{5}\right)^2 - 1\right]^2 - 1\end{aligned}$$
$$\begin{aligned}&= 2\left(\frac{18}{25} - 1\right)^2 - 1 \\ &= 2\left(-\frac{7}{25}\right)^2 - 1 \\ &= \frac{98}{625} - 1 \\ &= -\frac{527}{625}\end{aligned}$$

Example 4: If $\cos(x) = -\frac{3}{5}$, $\frac{\pi}{2} < x < \pi$, find the exact value of $\sin(2x)$.



$$\therefore \cos(x) = -\frac{3}{5}$$

$$\begin{aligned}\therefore \sin(x) &= \frac{\sqrt{5^2 - 3^2}}{5} \\ &= \frac{4}{5}\end{aligned}$$

$$\begin{aligned}\therefore \sin(2x) &= 2\sin(x)\cos(x) \\ &= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) \\ &= -\frac{24}{25}\end{aligned}$$

Example 5: Determine the exact value of $\cos\left(\frac{\pi}{8}\right)$.

$$\begin{aligned}\cos\left[2\left(\frac{\pi}{8}\right)\right] &= \cos\left(\frac{\pi}{4}\right) \\ 2\cos^2\left(\frac{\pi}{8}\right) - 1 &= \cos\left(\frac{\pi}{4}\right) \\ \cos\left(\frac{\pi}{8}\right) &= \pm \sqrt{\frac{\cos\left(\frac{\pi}{4}\right) + 1}{2}} \quad \leftarrow \text{since } \frac{\pi}{8} \text{ is in QI,} \right. \\ &\qquad\qquad\qquad \therefore \cos\left(\frac{\pi}{8}\right) > 0 \\ &= \sqrt{\frac{\frac{1}{\sqrt{2}} + 1}{2}} \\ &= \sqrt{\frac{1 + \sqrt{2}}{2\sqrt{2}}}\end{aligned}$$

Practice - SOLUTIONS

1. **Multiple Choice:** Select the best answer for each of the following.

C i. Given $\cos(\theta) = \frac{2}{3}$ in the first quadrant, the value of $\sin\left(\frac{\theta}{2}\right)$ is:

a. $\frac{\sqrt{3}}{2}$

b. $\frac{\sqrt{6}}{2}$

c. $\frac{\sqrt{6}}{6}$

d. $\frac{\sqrt{3}}{6}$

#1: $\cos \theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

Q1. positive $+\sqrt{\frac{\cos(\theta)-1}{-2}} = \sin\left(\frac{\theta}{2}\right)$

$\sqrt{\frac{\frac{2}{3}-1}{-2}} = \sin\left(\frac{\theta}{2}\right)$

$\frac{1}{\sqrt{6}} = \sin\left(\frac{\theta}{2}\right)$

C ii. $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ is equivalent to: $\sin 2\left(\frac{\pi}{8}\right) = \sin\left(\frac{\pi}{4}\right)$

a. $\sin\left(\frac{\pi}{8}\right)$

b. $\cos\left(\frac{\pi}{8}\right)$

c. $\sin\left(\frac{\pi}{4}\right)$

d. $\cos\left(\frac{\pi}{4}\right)$

B iii. The exact value of $1 - 2\sin^2\left(\frac{\pi}{8}\right)$ is: $= \cos 2\left(\frac{\pi}{8}\right) \Rightarrow \cos\left(\frac{\pi}{4}\right)$

a. $-\sqrt{2}$

b. $\frac{\sqrt{2}}{2}$

c. $-\frac{\sqrt{2}}{2}$

d. $\sqrt{2}$

2. Express as a single sine or cosine function.

a) $10\sin(x)\cos(x) = 5\sin(2x)$

b) $1 - 2\sin^2\left(\frac{2\theta}{3}\right) = \cos\left(\frac{4\theta}{3}\right)$

c) $5\sin(2x)\cos(2x) = \frac{5}{2}\sin(4x)$

d) $2\cos^2(5\theta) - 1 = \cos(10\theta)$

3. Simplify each expression.

a) $\frac{\sin(2a)}{\cos(a)} = \frac{2\sin(a)\cos(a)}{\cos(a)} = 2\sin(a)$

b) $2\tan(a)\cos^2(a) = 2 \frac{\sin(a)}{\cos(a)} \cdot \cos^2(a) = 2\sin(a)\cos(a) = \sin(2a)$

c) $2\sin^2(a) + \cos(2a) = 2\sin^2(a) + 1 - 2\sin^2(a) = 1$

4. Expand using a double angle formula.

b) $6\cos(6x)$

$= 6[2\cos^2(3x) - 1]$

$= 12\cos^2(3x) - 6$

or $6[1 - 2\sin^2(3x)]$

$= 6 - 12\sin^2(3x)$

or $6[\cos^2(3x) - \sin^2(3x)]$

$= 6\cos^2(3x) - 6\sin^2(3x)$

a) $3\sin(4x) = 6\sin(2x)\cos(2x)$

b) $6\cos(6x) =$

c) $1 - \cos(8x) =$

d) $\tan(4x) = \frac{2\tan(2x)}{1 + \tan^2(2x)}$

e) $\cos(2x) - \frac{\sin(2x)}{\sin(x)} = 2\cos^2(x) - 1 - \frac{2\sin(x)\cos(x)}{\sin(x)}$

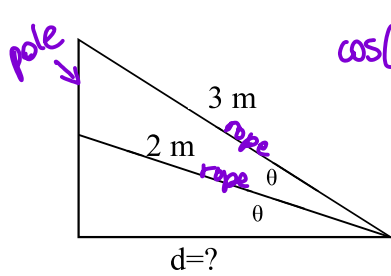
$= 2\cos^2(x) - 2\cos(x) - 1$

c) $1 - [1 - 2\sin^2(4x)]$
 $= 2\sin^2(4x)$

or $1 - [2\cos^2(4x) - 1]$
 $= -2\cos^2(4x)$

or $1 - [\cos^2(4x) - \sin^2(4x)]$
 $= \sin^2(4x) - \cos^2(4x)$

5. Two ropes (2m and 3m long) used to stabilize a pole for a volleyball net are anchored to the ground. The angle between the two ropes is equal to the angle between the ground and the lower rope. Determine the distance from the base of the pole to the point at which the ropes are anchored to the ground.



$$\cos(\theta) = \frac{d}{2} ; \cos(2\theta) = \frac{d}{3}$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

$$\frac{d}{3} = 2\left(\frac{d}{2}\right)^2 - 1$$

$$\frac{d}{3} = \frac{d^2}{2} - 1$$

$$0 = 3d^2 - 2d - 6$$

$$d = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-6)}}{2(3)}$$

$$= \frac{2 \pm 2\sqrt{19}}{6} \Rightarrow \frac{1 \pm \sqrt{19}}{3}$$

$$\text{or } \approx 1.78 \text{ m}$$

6. Express $\sin(2\theta)$ and $\cos(2\theta)$ in terms of $\tan(\theta)$.

7. Find the exact value of b .

$$\tan\left(\frac{\pi}{8}\right) = \frac{a}{12} \xrightarrow{\text{sub}} \frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)} = 1$$

$$\frac{2\left(\frac{a}{12}\right)}{1-\left(\frac{a}{12}\right)^2} = 1$$

$$\frac{a}{6} = 1 - \frac{a^2}{144}$$

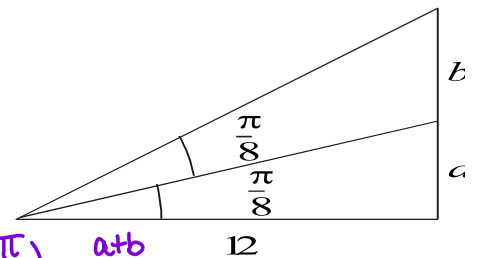
$$a^2 + 24a - 144 = 0$$

$$a = \frac{-24 \pm \sqrt{(24)^2 - 4(1)(-144)}}{2(1)}$$

$$= \frac{-24 \pm 24\sqrt{2}}{2}$$

$$= -12 \pm 12\sqrt{2}$$

$$\therefore a > 0 \therefore a = -12 + 12\sqrt{2}$$



$$\tan\left(\frac{\pi}{4}\right) = \frac{a+b}{12}$$

$$1 = \frac{a+b}{12}$$

$$12 = a+b$$

$$\text{sub } a = -12 + 12\sqrt{2}$$

$$12 = (-12 + 12\sqrt{2}) + b$$

$$\therefore b = 24 - 12\sqrt{2}$$

$$6. \sin(2\theta) = 2\sin(\theta)\cos(\theta) \cdot \frac{\cos(\theta)}{\cos(\theta)}$$

$$= 2\tan(\theta)\cos^2(\theta)$$

$$= 2\tan(\theta) \cdot \frac{1}{\sec^2(\theta)}$$

$$= 2\tan(\theta) \cdot \frac{1}{1+\tan^2(\theta)}$$

$$= \frac{2\tan(\theta)}{1+\tan^2(\theta)}$$

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$= \frac{1}{\sec^2(\theta)} - \frac{1}{\csc^2(\theta)}$$

$$= \frac{1}{1+\tan^2(\theta)} - \frac{1}{\cot^2(\theta)+1}$$

$$= \frac{1}{1+\tan^2(\theta)} - \frac{1}{\frac{1}{\tan^2(\theta)}+1}$$

$$= \frac{1}{1+\tan^2(\theta)} - \frac{1}{\frac{1+\tan^2(\theta)}{\tan^2(\theta)}}$$

$$= \frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$$

Practice-Double Angle Formulas-Solution

Practice

1. **Multiple Choice:** Select the best answer for each of the following

— **C** — i. Given $\cos(\theta) = \frac{2}{3}$ in the first quadrant, the value of $\sin\left(\frac{\theta}{2}\right)$ is:

- a. $\frac{\sqrt{3}}{2}$ b. $\frac{\sqrt{6}}{2}$ c. $\frac{\sqrt{6}}{6}$ d. $\frac{\sqrt{3}}{6}$

— **C** — ii. $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ is equivalent to :

- a. $\sin\left(\frac{\pi}{8}\right)$ b. $\cos\left(\frac{\pi}{8}\right)$ c. $\sin\left(\frac{\pi}{4}\right)$ d. $\cos\left(\frac{\pi}{4}\right)$

— **B** — iii. The exact value of $1 - 2\sin^2\left(\frac{\pi}{8}\right)$ is:

- a. $-\sqrt{2}$ b. $\frac{\sqrt{2}}{2}$ c. $-\frac{\sqrt{2}}{2}$ d. $\sqrt{2}$

2. Express as a single sine or cosine function.

a) $10\sin(x)\cos(x) = \mathbf{5\sin(2x)}$

b) $1 - 2\sin^2\left(\frac{2\theta}{3}\right) = \mathbf{\cos\left(\frac{4\theta}{3}\right)}$

c) $5\sin(2x)\cos(2x) = \mathbf{\frac{5}{2}\sin(4x)}$

d) $2\cos^2(5\theta) - 1 = \mathbf{\cos(10\theta)}$

3. Simplify each expression.

a) $\frac{\sin 2a}{\cos a} = \mathbf{2\sin a}$

b) $2\tan a \cos^2 a = \mathbf{\sin(2a)}$

c) $2\sin^2 a + \cos 2a = \mathbf{1}$

4. Expand using a double angle formula.

a) $3\sin(4x) = \mathbf{6\sin(2x)\cos(2x)}$

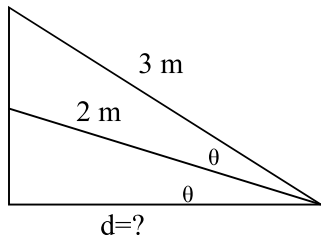
b) $6\cos(6x) = \mathbf{12\cos^2(3x) - 6}$

c) $1 - \cos(8x) = \mathbf{2\sin^2(4x)}$

d) $\tan(4x) = \frac{\mathbf{2\tan(2x)}}{\mathbf{1 + \tan^2(2x)}}$

e) $\cos(2x) - \frac{\sin(2x)}{\sin(x)} = \mathbf{2\cos^2(x) - 2\cos(x) - 1}$

5. Two ropes (2m and 3m long) used to stabilize a pole for a volleyball net are anchored to the ground. The angle between the two ropes is equal to the angle between the ground and the lower rope. Determine the distance from the base of the pole to the point at which the ropes are anchored to the ground.



$$\cos \theta = \frac{d}{2}$$

$$\cos(2\theta) = \frac{d}{3}$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\frac{d}{3} = 2\left(\frac{d}{2}\right)^2 - 1$$

$$3d^2 - 2d - 6 = 0$$

$$d = \frac{1 + \sqrt{19}}{3}$$

$$d \doteq 1.78 \text{ m}$$

6. Express $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan \theta$.

$$\sin 2\theta = 2\sin\theta\cos\theta \times \frac{\cos\theta}{\cos\theta}$$

$$= 2 \frac{\cos\theta}{\cos\theta} \times \cos^2\theta$$

$$= 2\tan\theta \times \frac{1}{1 + \tan^2\theta}$$

$$= \frac{2\tan\theta}{1 + \tan^2\theta}$$

7. Find the exact value of b .

$$\tan\left(\frac{\pi}{8}\right) = \frac{a}{12}$$

$$\tan\left(\frac{\pi}{8} + \frac{\pi}{8}\right) = \frac{a+b}{12} \Rightarrow 1 = \frac{a+b}{12}$$

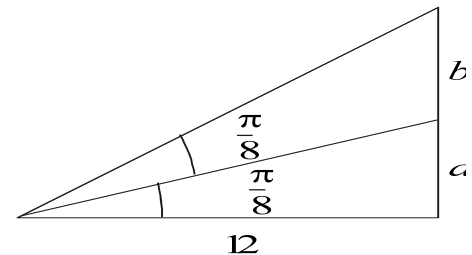
$$a + b = 12$$

$$1 = \frac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)} \Rightarrow 1 = \frac{2\left(\frac{a}{12}\right)}{1 - \left(\frac{a}{12}\right)^2}$$

$$1 = \frac{24a}{144 - a^2} \Rightarrow a^2 + 24a - 144 = 0$$

$$a = 12(\sqrt{2} - 1)$$

$$b = 12(2 - \sqrt{2})$$



Warm Up

1. The expression $4\cos^2\left(\frac{3}{2}\theta\right) - 2$ expressed as a single trig function is:

$$2[2\cos^2\left(\frac{3}{2}\theta\right) - 1] = 2\cos(3\theta)$$

- A. $\cos\left(\frac{3}{2}\theta\right)$ B. $2\cos(2\theta)$ C. $2\cos(3\theta)$ D. $2\cos\left(\frac{3}{2}\theta\right)$

2. The exact value of the $\frac{2\tan\left(\frac{\pi}{8}\right)}{1 - \tan^2\left(\frac{\pi}{8}\right)}$ is: $\Rightarrow \tan\left(\frac{\pi}{4}\right) = 1$

- A. $\frac{\sqrt{3}}{3}$ B. $\sqrt{3}$ C. 1 D. $\frac{\sqrt{2}}{2}$

3. If $4\tan(2\theta) - 3 = 0, 0 < \theta < \frac{\pi}{4}$, and $13\sin(\alpha) + 5 = 0, 0 < \alpha < \frac{3\pi}{2}$, determine the **exact** value of $\tan(\theta) + \sin(2\alpha)$.

$$\tan(2\theta) = \frac{3}{4}, 0 < \theta < \frac{\pi}{4}$$

$$\frac{2\tan(\theta)}{1 - \tan^2(\theta)} = \frac{3}{4}$$

$$8\tan(\theta) = 3 - 3\tan^2(\theta)$$

$$3\tan^2(\theta) + 8\tan(\theta) - 3 = 0$$

$$[3\tan(\theta) - 1][\tan(\theta) + 3] = 0$$

$$\tan(\theta) = \frac{1}{3} \quad \downarrow \quad \tan(\theta) = -3$$

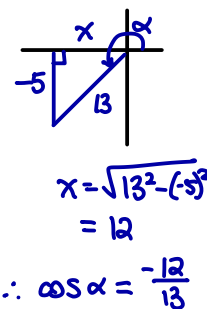
is inadmissible
 $\therefore 0 < \theta < \frac{\pi}{4}$

$$\sin(\alpha) = -\frac{5}{13}, 0 < \alpha < \frac{3\pi}{2}$$

$$\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$$

$$= 2\left(-\frac{5}{13}\right)\left(-\frac{12}{13}\right)$$

$$= \frac{120}{169}$$



$$\therefore \tan(\theta) + \sin(2\alpha) = \left(\frac{1}{3}\right) + \left(\frac{120}{169}\right) = \frac{529}{507}$$

4. Determine the **exact** value of $\cos^2\left(\frac{7\pi}{8}\right)$.

$$\cos\left(\frac{7\pi}{4}\right) = \cos\left[2\left(\frac{7\pi}{8}\right)\right]$$

$$\cos\left(\frac{7\pi}{4}\right) = 2\cos^2\left(\frac{7\pi}{8}\right) - 1$$

$$\frac{\cos\left(\frac{7\pi}{4}\right) + 1}{2} = \cos^2\left(\frac{7\pi}{8}\right)$$

$$\frac{\cos\left(\frac{\pi}{4}\right) + 1}{2} = \cos^2\left(\frac{7\pi}{8}\right)$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1\right) = \cos^2\left(\frac{7\pi}{8}\right)$$

$$\frac{\sqrt{2} + 2}{4} = \cos^2\left(\frac{7\pi}{8}\right)$$

5.1 Part II- Proving Trigonometric Identities

Fill in the blanks with the words in the box.

Counter-example	trig-identity identity	equal
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- A statement of equality between two expressions that is true for all values of the variables for which the expressions are defined is called a(n) identity.
- An identity involving trigonometric expressions is called a trig identity.
- Our goal is to prove that one side of an expression is equal to the other side of the expression.
- A counter-example can be used to show that an equation is not an identity

Strategies for Proving Trig Identities:

- Write everything in terms of sine and cosine
- Be aware of equivalent forms of the fundamental identities, ie: $\sin^2(\theta) + \cos^2(\theta) = 1$ has an alternative form: $\sin^2(\theta) = 1 - \cos^2(\theta)$
- Try to rewrite the more complicated side of the equation so that it is identical to the simpler side.
- Usually any factoring or indicated algebraic operations should be performed, ie:

$$1 + \sin(2x) = (\sin(x) + \cos(x))^2 \quad \text{or}$$

$$\begin{aligned} \sin^3(x) + \cos^3(x) &= (\sin(x) + \cos(x))(\sin^2(x) - \sin(x)\cos(x) + \cos^2(x)) \\ &= (\sin(x) + \cos(x))\left(1 - \frac{1}{2}\sin(2x)\right) \end{aligned}$$

- If an expression contains $1 \pm \sin x$, $\sec x \pm \tan x$ or $\csc x \pm \cot x$ multiplying both numerator and denominator by $1 \mp \sin x$, $\sec x \mp \tan x$ or $\csc x \mp \cot x$ would give $\cos^2 x$, 1 or -1.

1. Prove the following identities:

a) $\frac{1 + \sec(x)}{\tan(x) + \sin(x)} = \csc(x)$

$$\begin{aligned} \text{L.S.} &= \frac{1 + \sec(x)}{\tan(x) + \sin(x)} \\ &= \frac{1 + \frac{1}{\cos(x)}}{\frac{\sin(x)}{\cos(x)} + \sin(x)} \\ &= \frac{\frac{\cos(x) + 1}{\cos(x)}}{\frac{\sin(x) + \sin(x)\cos(x)}{\cos(x)}} \\ &= \frac{\cos(x) + 1}{\sin(x) + \sin(x)\cos(x)} \\ &= \frac{\cos(x) + 1}{\sin(x)[1 + \cos(x)]} \\ &= \frac{1}{\sin(x)} \\ &= \csc(x) \\ &= \text{R.S.} \\ &\therefore \text{L.S.} = \text{R.S.} \\ &\therefore \text{The identity is true} \end{aligned}$$

b) $\sin(2A) = \frac{2\tan(A)}{1 + \tan^2(A)}$

$$\begin{aligned} \text{R.S.} &= \frac{2\tan(A)}{1 + \tan^2(A)} \\ &= \frac{2\sin(A)}{\cos(A)} \cdot \frac{\cos^2(A)}{\cos^2(A) + \sin^2(A)} \\ &= \frac{2\sin(A)\cos(A)}{1} \\ &= \sin(2A) \\ &= \text{L.S.} \\ &\therefore \text{L.S.} = \text{R.S.} \\ &\therefore \text{The identity is true} \end{aligned}$$

$$c) \cos^4(x) - \sin^4(x) = \cos(2x)$$

$$\begin{aligned} \text{L.S.} &= \cos^4(x) - \sin^4(x) \\ &= [\cos^2(x) + \sin^2(x)][\cos^2(x) - \sin^2(x)] \\ &= (1)[\cos^2(x) - \sin^2(x)] \\ &= \cos(2x) \\ &= \text{R.S.} \end{aligned}$$

$$\therefore \text{L.S.} = \text{R.S.}$$

$\therefore$ The identity is true

$$d) \cos^4(A) + \sin^4(A) = 1 - \frac{1}{2} \sin^2(2A)$$

$$\begin{aligned} \text{L.S.} &= \cos^4(A) + \sin^4(A) \\ &= \cos^4(A) + 2\sin^2(A)\cos^2(A) + \sin^4(A) - 2\sin^2(A)\cos^2(A) \\ &= [\cos^2(A) + \sin^2(A)]^2 - 2\sin^2(A)\cos^2(A) \\ &= 1 - \frac{1}{2} [2\sin(A)\cos(A)]^2 \\ &= 1 - \frac{\sin^2(2A)}{2} \\ &= \text{R.S.} \end{aligned}$$

$$\therefore \text{L.S.} = \text{R.S.}$$

$\therefore$ The identity is true.

$$e) \frac{1 - \sin(2x)}{\cos(2x)} = \frac{1 - \tan(x)}{1 + \tan(x)}$$

$$\begin{aligned} \text{R.S.} &= \frac{1 - \tan(x)}{1 + \tan(x)} \\ &= \frac{1 - \frac{\sin(x)}{\cos(x)}}{1 + \frac{\sin(x)}{\cos(x)}} \\ &= \frac{\frac{\cos(x) - \sin(x)}{\cos(x)}}{\frac{\cos(x) + \sin(x)}{\cos(x)}} \\ &= \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} \times \frac{\cos(x) - \sin(x)}{\cos(x) - \sin(x)} \\ &= \frac{\cos^2(x) - 2\sin(x)\cos(x) + \sin^2(x)}{\cos^2(x) - \sin^2(x)} \\ &= \frac{1 - \sin(2x)}{\cos(2x)} \\ &= \text{L.S.} \end{aligned}$$

$$\therefore \text{L.S.} = \text{R.S.}$$

$\therefore$ The identity is true

$$f) 2\cos^3(x) + \sin(2x)\sin(x) = 2\cos(x)$$

$$\begin{aligned} \text{L.S.} &= 2\cos^3(x) + \sin(2x)\sin(x) \\ &= 2\cos^3(x) + 2\sin(x)\cos(x)\sin(x) \\ &= 2\cos^3(x) + 2\sin^2(x)\cos(x) \\ &= 2\cos(x)[\cos^2(x) + \sin^2(x)] \\ &= 2\cos(x) \\ &= \text{R.S.} \end{aligned}$$

$$\therefore \text{L.S.} = \text{R.S.}$$

$\therefore$ The identity is true

PRACTICE

1. Prove each identity.

a) $\frac{\sec(\theta) - 1}{1 - \cos(\theta)} = \sec(\theta)$

h) $\frac{1 + \tan(A)}{\sin(A)} - \sec(A) = \csc(A)$

b) $\tan(x) + \cot(x) = \frac{2}{\sin(2x)}$

i) $\frac{\sin(t) - \cos(t)}{\cos(t)} + \frac{\sin(t) + \cos(t)}{\sin(t)} = \sec(t)\csc(t)$

c) $\frac{1 + \tan^2(x)}{1 + \cot^2(x)} = \frac{1 - \cos^2(x)}{\cos^2(x)}$

j) $\frac{[1 + \cos(2x)][1 + \cos(x)]}{[\sin(2x)][\cos(x)]} = \cot\left(\frac{x}{2}\right)$

d) $\frac{1}{1 + \sec(\theta)} + \frac{1}{1 - \sec(\theta)} = -2\cot^2(\theta)$

k) $\frac{4 - \sin^2(2x)}{4\cos^4(x)} = \tan^4(x) + \tan^2(x) + 1$

e) $\frac{1 + \sec(x)}{\tan(x) + \sin(x)} = \csc(x)$

l) $1 - \sin(x)\cos(x) = \frac{\sin^2(x)}{1 + \cot(x)} + \frac{\cos^2(x)}{1 + \tan(x)}$

f) $\cos(a + b)\cos(a - b) = \cos^2(a) - \sin^2(b)$

m) $\frac{\sin(x - y)}{\sin(x)\sin(y)} = \cot(y) - \cot(x)$

g) $\frac{\cos(x) - \sin(y)}{\cos(y) - \sin(x)} = \frac{\cos(y) + \sin(x)}{\cos(x) + \sin(y)}$

n) $\frac{\sec(x) + \tan(x)}{\sin(x)} = \frac{\csc(x)}{\sec(x) - \tan(x)}$

2*. If $2\cos^2(x) + 4\sin(x)\cos(x)$ is expressed in the form $A\sin(2x) + B\cos(2x) + C$ where $A, B, C \in \mathbb{R}$, determine the values of A, B, and C.

3*. Write $2\sin(2x) + \sqrt{12}\cos(2x)$ in the form $y = A\cos(2x - \theta)$ by finding $A > 0$ and $\theta \in [0, 2\pi]$.

5.1 Part III Solving Trigonometric Equations

What is a trig equation?

- A trig equation is an equation that contains one or more trigonometric functions.

How is it similar to a trig identity?

- A trig equation *can* be but *does not have to* be a trig identity
 - A trig identity is an equation that is true for **all values** of the variable for which expressions of both sides of the equation are defined
- A trig equation that is not an identity is only true for **certain values** of the variable.

What does it mean to **SOLVE** a trig equation?

- Much like solving a linear equation, we are looking for **all of the values of the variable that makes the equation true.**

Ex.1) Find the exact solutions for $2\cos(x) + \sqrt{3} = 0$ for $x \in [0, 2\pi]$.

$$2\cos(x) + \sqrt{3} = 0$$

$$\cos(x) = \frac{-\sqrt{3}}{2}$$

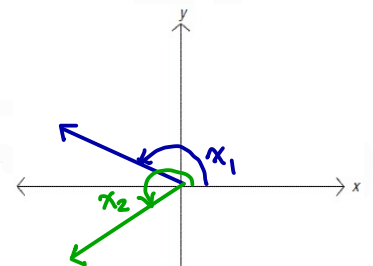
$$\therefore \alpha = \frac{\pi}{6}$$

$\therefore \cos$ is -ve in Q2 & Q3

$$\therefore x_1 = \pi - \frac{\pi}{6} \quad \text{or} \quad x_2 = \pi + \frac{\pi}{6}$$

$$= \frac{5\pi}{6} \quad \quad \quad = \frac{7\pi}{6}$$

$$\therefore x = \frac{5\pi}{6}, \frac{7\pi}{6}$$



Ex.2) Find the exact solutions for $2\sin(2x) = \sqrt{3}$ for $x \in [0, 2\pi]$. Method ①: Double the domain $\rightarrow 2x \in [0, 4\pi]$

$$2\sin(2x) = \sqrt{3}$$

$$\sin(2x) = \frac{\sqrt{3}}{2}$$

$$\therefore \alpha = \frac{\pi}{3}$$

$\therefore \sin$ is +ve in Q1 & Q2

$$\therefore 2x_1 = \frac{\pi}{3} \quad \text{or} \quad 2x_2 = \pi - \frac{\pi}{3}$$

$$x_1 = \frac{\pi}{6} \quad \quad \quad 2x_2 = \frac{2\pi}{3}$$

$$x_2 = \frac{\pi}{3}$$

Method ①: Add periods of 2π to $2x_1$ and $2x_2$:

$$2x_3 = \frac{\pi}{3} + 2\pi$$

$$2x_4 = \frac{2\pi}{3} + 2\pi$$

$$2x_3 = \frac{7\pi}{3}$$

$$2x_4 = \frac{8\pi}{3}$$

$$x_3 = \frac{7\pi}{6}$$

$$x_4 = \frac{4\pi}{3}$$

Method ②: Add periods of π to x_1 and x_2 :

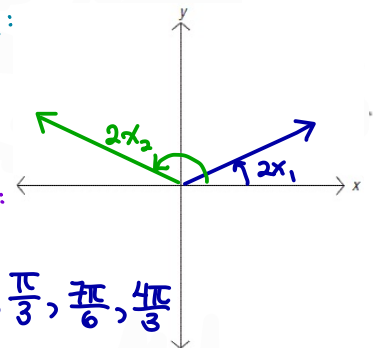
$$x_3 = \frac{\pi}{6} + \pi$$

$$= \frac{7\pi}{6}$$

$$x_4 = \frac{\pi}{3} + \pi$$

$$= \frac{4\pi}{3}$$

$$\therefore x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$$



Ex.3) Find the exact solutions for $3\tan^2(\theta) = 1$ for $\theta \in [0, 2\pi]$.

$$3\tan^2(\theta) = 1$$

$$\tan^2(\theta) = \frac{1}{3}$$

$$\tan(\theta) = \pm \frac{1}{\sqrt{3}}$$

$$\therefore \alpha = \frac{\pi}{6}$$

$\therefore \tan$ is +ve and -ve, consider all quadrants

$$\therefore \theta_1 = \frac{\pi}{6} \quad \text{or} \quad \theta_2 = \pi - \frac{\pi}{6}$$

$$= \frac{5\pi}{6}$$

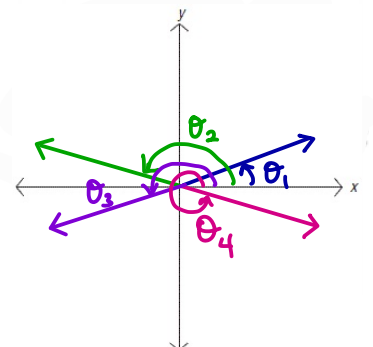
$$\theta_3 = \pi + \frac{\pi}{6}$$

$$= \frac{7\pi}{6}$$

$$\theta_4 = 2\pi - \frac{\pi}{6}$$

$$= \frac{11\pi}{6}$$

$$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$



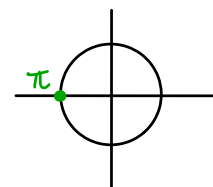
Ex.4) Find the exact solutions for $\tan^2(\theta) + 3\sec(\theta) + 3 = 0, 0 \leq \theta \leq 2\pi$

$$\begin{aligned}\tan^2(\theta) + 3\sec(\theta) + 3 &= 0 \\ \sec^2(\theta) - 1 + 3\sec(\theta) + 3 &= 0 \\ \sec^2(\theta) + 3\sec(\theta) + 2 &= 0 \\ [\sec(\theta) + 2][\sec(\theta) + 1] &= 0\end{aligned}$$

$$\begin{aligned}\therefore \sec(\theta) + 2 &= 0 \\ \sec(\theta) &= -2 \\ \cos(\theta) &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}\alpha &= \frac{\pi}{3} \\ \therefore \cos \text{ is -ve in Q2 \& Q3,} \\ \therefore \theta_1 &= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}\end{aligned}$$

$$\begin{aligned}\text{or } \sec(\theta) + 1 &= 0 \\ \sec(\theta) &= -1 \\ \cos(\theta) &= -1 \\ \theta_3 &= \pi\end{aligned}$$



$$\therefore \theta = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$$

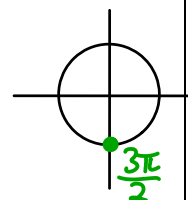
Ex.5) Find the solutions for $3\cos^2(\theta) - \sin(\theta) - 1 = 0, 0 \leq \theta \leq 2\pi$

$$\begin{aligned}3\cos^2(\theta) - \sin(\theta) - 1 &= 0 \\ 3[1 - \sin^2(\theta)] - \sin(\theta) - 1 &= 0 \\ 3 - 3\sin^2(\theta) - \sin(\theta) - 1 &= 0 \\ 3\sin^2(\theta) + \sin(\theta) - 2 &= 0 \\ [3\sin(\theta) - 2][\sin(\theta) + 1] &= 0\end{aligned}$$

$$\begin{aligned}\therefore 3\sin(\theta) - 2 &= 0 \\ \sin(\theta) &= \frac{2}{3} \\ \therefore \alpha &\doteq 0.730\end{aligned}$$

$$\begin{aligned}\therefore \sin \text{ is +ve in Q1 \& Q2,} \\ \therefore \theta_1 &= 0.730 \text{ or } \theta_2 = \pi - 0.730 \\ &= 2.41\end{aligned}$$

$$\begin{aligned}\text{or } \sin(\theta) + 1 &= 0 \\ \sin(\theta) &= -1 \\ \theta_3 &= \frac{3\pi}{2}\end{aligned}$$



$$\therefore \theta = \frac{3\pi}{2}, 0.730, 2.41$$

Ex.6) Solve the equation $\sin(4x) - \cos(2x) = 0, x \in [0, 2\pi]$

$$\begin{aligned}\sin(4x) - \cos(2x) &= 0 \\ 2\sin(2x)\cos(2x) - \cos(2x) &= 0 \\ \cos(2x)[2\sin(2x) - 1] &= 0\end{aligned}$$

Change the domain: $(2x) \in [0, 4\pi]$

$$\therefore \cos(2x) = 0$$

$$2x_1 = \frac{\pi}{2} \text{ or } 2x_2 = \frac{3\pi}{2}$$

$$x_1 = \frac{\pi}{4}$$

$$x_2 = \frac{3\pi}{4}$$

$$\begin{aligned}\text{or } 2\sin(2x) - 1 &= 0 \\ \sin(2x) &= \frac{1}{2}\end{aligned}$$

$$\therefore \alpha = \frac{\pi}{6}$$

$\therefore \sin$ is +ve in Q1 and Q2,

$$\therefore 2x_5 = \frac{\pi}{6}$$

$$x_5 = \frac{\pi}{12}$$

$$2x_6 = \frac{5\pi}{6}$$

$$x_6 = \frac{5\pi}{12}$$

$$\begin{aligned}x_7 &= \frac{\pi}{12} + \pi \\ &= \frac{13\pi}{12}\end{aligned}$$

$$\begin{aligned}x_8 &= \frac{5\pi}{12} + \pi \\ &= \frac{17\pi}{12}\end{aligned}$$

Add 2π to $2x_1$ and $2x_2$ to find coterminal angles

$$2x_3 = \frac{\pi}{2} + 2\pi \text{ or } 2x_4 = \frac{3\pi}{2} + 2\pi$$

$$2x_3 = \frac{5\pi}{2}$$

$$x_3 = \frac{5\pi}{4}$$

$$2x_4 = \frac{7\pi}{2}$$

$$x_4 = \frac{7\pi}{4}$$

Alternate method:
Add a period to x_1, x_2 , etc. to find all solutions in given domain

$$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$$

Ex.7) Solve $\sin(2x) - \cos(2x) = 0$, $0 \leq x \leq 2\pi$

$$\sin(2x) - \cos(2x) = 0$$

$$\sin(2x) = \cos(2x)$$

$$\tan(2x) = 1$$

$$\therefore \alpha = \frac{\pi}{4}$$

$\therefore \tan$ is +ve in Q1 & Q3,

$$\therefore 2x_1 = \frac{\pi}{4} \quad \text{or} \quad 2x_2 = \pi + \frac{\pi}{4}$$

$$x_1 = \frac{\pi}{8}$$

$$2x_2 = \frac{5\pi}{4}$$

$$x_2 = \frac{5\pi}{8}$$

Add 2π to $2x_1$ and $2x_3$ to find coterminal angles.

$$2x_3 = \frac{\pi}{4} + 2\pi$$

$$2x_3 = \frac{9\pi}{4}$$

$$x_3 = \frac{9\pi}{8}$$

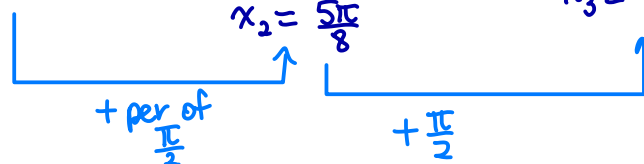
$$2x_4 = \frac{5\pi}{4} + 2\pi$$

$$2x_4 = \frac{13\pi}{4}$$

$$x_4 = \frac{13\pi}{8}$$

Alternate method:

Add periods to x_1 , x_2 , etc to find all solutions in the domain



$$\therefore x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$$

Ex.8) Solve $4\cos(2x) - \sin(x)\csc^3(x) + 2 = 0$, $0 \leq x \leq 2\pi$

$$4\cos(2x) - \sin(x)\csc^3(x) + 2 = 0$$

$$4[1 - 2\sin^2(x)] - \frac{\sin(x)}{\sin^3(x)} + 2 = 0$$

$$4 - 8\sin^2(x) - \frac{1}{\sin^2(x)} + 2 = 0$$

$$8\sin^4(x) - 6\sin^2(x) + 1 = 0$$

$$[4\sin^2(x) - 1][2\sin^2(x) - 1] = 0$$

$$\therefore 4\sin^2(x) - 1 = 0$$

$$\sin(x) = \pm \frac{1}{2}$$

$$\therefore \alpha = \frac{\pi}{6}$$

$\therefore \sin$ is +ve and -ve, consider all quadrants

$$x_1 = \frac{\pi}{6} \quad x_2 = \frac{5\pi}{6}$$

$$x_3 = \frac{7\pi}{6} \quad x_4 = \frac{11\pi}{6}$$

$$2\sin^2(x) - 1 = 0$$

$$\sin(x) = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \alpha = \frac{\pi}{4}$$

$\therefore \sin$ is +ve and -ve, consider all quadrants

$$\therefore x_5 = \frac{\pi}{4} \quad x_6 = \frac{3\pi}{4}$$

$$x_7 = \frac{5\pi}{4} \quad x_8 = \frac{7\pi}{4}$$

$$\therefore x = \frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{6}$$

Practice

1. Solve for x on the interval $[0, 2\pi]$.

a) $6\sin^2(x) - \sin(x) - 1 = 0$ b) $\cot(x)\cos^2(x) = \cot(x)$ c)

$4\tan(x) - \sec^2(x) = 0$

d) $2\sin^2(x) - \sin(x) - 1 = 0$ e) $4\sin^3(x) + 2\sin^2(x) - 2\sin(x) - 1 = 0$ f)

$\sin(x) + \sqrt{2} = -\sin(x)$

g) $2\cos(3x - 1) = 0$

h) $\sin(2x)\cos(x) - \cos(2x)\sin(x) = 0$ i)

$\sec^2(x) - 2\tan(x) = 4$

j) $3\tan\left(\frac{x}{2}\right) + 3 = 0$

k) $-6\sin(2x)\cos(x) + 8\cos(2x) + 3\sin(x) + 4 = 0$

l) $\cot(x)\cos^2(x) = 2\cot(x)$ m) $\frac{1 + \sin(x)}{\cos(x)} + \frac{\cos(x)}{1 + \sin(x)} = 4$ n) $2\sin^2(x) + 3\cos(x) - 3 = 0$

o) $2\sin(x)\tan(x) - \tan(x) - 2\sin(x) + 1 = 0$

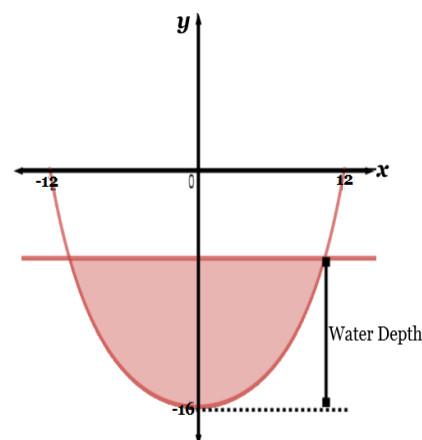
p) $\cos(x)\tan(x) - 1 + \tan(x) - \cos(x) = 0$

2. A weight hanging from a spring is set in motion moving up and down. Its distance, d , (in cm) above or below its “rest” position is described by $d(t) = 5(\sin(6t) - 4\cos(6t))$. At what times during the first 2 seconds is the weight at the rest position ($d=0$).

3. The diagram below shows the boundary of the cross section of a water channel. The equation that represents this boundary is

$$y = 16\sec\left(\frac{\pi x}{36}\right) - 32 \text{ where } x \text{ and } y \text{ are both measured in}$$

cm. Find the width of the water surface when the water depth in the channel is 10 cm. [Round your answer to 2dp].



4. Solve $\frac{\sin^2(x) - 3\sin(x)}{1 - 2\cos(x)} \geq 0$ over the interval $0 < x < 2\pi$.

5. Solve $\sin x = \frac{x}{2}$, $0 \leq x \leq 2.5$, $x \in \mathbb{R}$ (Use GDC).

Solving Trigonometric Equations Practice Solutions

Practice

1. Solve for x on the interval $[0, 2\pi]$.

a) $6\sin^2(x) - \sin(x) - 1 = 0$

$$(2\sin(x) - 1)(3\sin(x) + 1) = 0$$

$$\sin(x) = \frac{1}{2} \quad \text{or} \quad \sin(x) = -\frac{1}{3}$$

$$x = \sin^{-1}\left(\frac{1}{2}\right) \quad \text{or} \quad x = \sin^{-1}\left(-\frac{1}{3}\right)$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = 3.48\text{rad}, 5.94\text{rad}$$

b) $\cot(x)\cos^2(x) = \cot(x)$

$$\cot(x)\cos^2(x) - \cot(x) = 0 \quad \text{Res. } \sin(x) \neq 0$$

$$\cot(x)[\cos^2(x) - 1] = 0 \quad x \neq k\pi$$

$$\cot(x) = 0 \quad \cos^2(x) - 1 = 0$$

$$\cos(x) = 0 \quad \cos(x) = \pm 1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$x = 0, 2\pi, \pi$$

inadmissible

c) $4\tan(x) - \sec^2(x) = 0 \quad \text{Res. } \cos(x) \neq 0$

$$4\tan(x) - (1 + \tan^2(x)) = 0$$

$$\tan^2(x) - 4\tan(x) + 1 = 0$$

$$\tan(x) = \frac{4 \pm \sqrt{4^2 - 4(1)(1)}}{2}$$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

$$x = \tan^{-1}(2 + \sqrt{3}) \quad \text{or} \quad x = \tan^{-1}(2 - \sqrt{3})$$

$$x = \frac{5\pi}{12}$$

$$x = \frac{\pi}{12}$$

d) $2\sin^2(x) - \sin(x) - 1 = 0$

$$(2\sin^2(x) + 1)(\sin(x) - 1) = 0$$

$$\sin(x) = -\frac{1}{2} \quad \text{or} \quad \sin(x) = 1$$

$$x = 2k\pi - \frac{\pi}{6} \quad x = 2k\pi + \frac{\pi}{2}$$

$$x = 2k\pi + \frac{7\pi}{6}$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{2}$$

e) $4\sin^3(x) + 2\sin^2(x) - 2\sin(x) - 1 = 0$

$$[4\sin^3(x) + 2\sin^2(x)] - [2\sin(x) + 1] = 0$$

$$2\sin^2(x)[2\sin(x) + 1] - [2\sin(x) + 1] = 0$$

$$[2\sin(x) + 1][2\sin^2(x) - 1] = 0$$

$$\sin(x) = -\frac{1}{2} \quad \text{or} \quad \sin^2(x) = \frac{1}{2} \rightarrow \sin(x) = \frac{\sqrt{2}}{2} \quad \text{or} \quad \sin(x) = -\frac{\sqrt{2}}{2}$$

$$x = 2k\pi - \frac{\pi}{6}$$

$$x = 2k\pi + \frac{\pi}{4}$$

$$x = 2k\pi - \frac{\pi}{4}$$

$$x = 2k\pi + \frac{7\pi}{6}$$

$$x = 2k\pi + \frac{3\pi}{4}$$

$$x = 2k\pi + \frac{5\pi}{4}$$

$$x = \frac{11\pi}{6}, \frac{7\pi}{6}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$x = \frac{7\pi}{4}, \frac{5\pi}{4}$$

Solving Trigonometric Equations Practice Solutions

f) $\sin(x) + \sqrt{2} = -\sin(x)$

$$2\sin(x) = -\sqrt{2}$$

$$\sin(x) = \frac{-\sqrt{2}}{2}$$

$$x = 2k\pi - \frac{\pi}{4}$$

$$x = 2k\pi + \frac{5\pi}{4}$$

$$x = \frac{7\pi}{4}, \frac{5\pi}{4}$$

g) $2\cos(3x-1) = 0$

$$2\cos(3x-1) = 0$$

$$3x-1 = (2k-1)\frac{\pi}{2}$$

$$3x = (2k-1)\frac{\pi}{2} + 1$$

$$x = (2k-1)\frac{\pi}{6} + \frac{1}{3}$$

$$x = 1.38 \text{ rad}, 1.90 \text{ rad}, 2.95 \text{ rad}, 3.99 \text{ rad}, 5.04 \text{ rad}, 6.09 \text{ rad}$$

h) $\sin(2x)\cos(x) - \cos(2x)\sin(x) = 0$

$$\sin(2x-x) = 0$$

$$\sin(x) = 0$$

$$x = k\pi$$

$$x = 0, \pi, 2\pi$$

i) $\sec^2(x) - 2\tan(x) = 4$

$$1 + \tan^2(x) - 2\tan(x) = 4$$

$$\tan^2(x) - 2\tan(x) - 3 = 0$$

$$(\tan(x) - 3)(\tan(x) + 1) = 0$$

$$\tan(x) = 3 \text{ or } \tan(x) = -1$$

$$x = k\pi + 1.25 \text{ or } x = k\pi - \frac{\pi}{4}$$

j) $3\tan\left(\frac{x}{2}\right) + 3 = 0$

$$\tan\left(\frac{x}{2}\right) = -1$$

$$\frac{x}{2} = k\pi - \frac{\pi}{4}$$

$$x = 2k\pi - \frac{\pi}{2}$$

$$x = \frac{3\pi}{2}$$

$$x = 1.25, 4.39 \text{ or } x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Solving Trigonometric Equations Practice Solutions

$$k) -6\sin(2x)\cos(x) + 8\cos(2x) + 3\sin(x) + 4 = 0$$

$$-6\sin(2x)\cos(x) + 8\cos(2x) + 3\sin(x) + 4 = 0$$

$$-12\sin(x)\cos^2(x) + 8(1 - 2\sin^2(x)) + 3\sin(x) + 4 = 0$$

$$-12\sin(x)(1 - \sin^2(x)) + 8 - 16\sin^2(x) + 3\sin(x) + 4 = 0$$

$$-12\sin(x) + 12\sin^3(x) + 8 - 16\sin^2(x) + 3\sin(x) + 4 = 0$$

$$12\sin^3(x) - 16\sin^2(x) - 9\sin(x) + 12 = 0$$

$$4\sin^2(x)[3\sin(x) - 4] - 3[3\sin(x) - 4] = 0$$

$$[3\sin(x) - 4][4\sin^2(x) - 3] = 0$$

$$\underbrace{\sin(x) = \frac{4}{3}}_{\text{inadmissible}} \text{ or } 4\sin^2(x) = 3$$

$$\sin^2(x) = \frac{3}{4}$$

$$\sin(x) = \frac{\sqrt{3}}{2}$$

$$x = 2k\pi + \frac{\pi}{3}$$

$$x = 2k\pi + \frac{2\pi}{3}$$

$$\boxed{x = \frac{\pi}{3}, \frac{2\pi}{3}}$$

$$\sin(x) = -\frac{\sqrt{3}}{2}$$

$$x = 2k\pi - \frac{\pi}{3}$$

$$x = 2k\pi + \frac{4\pi}{3}$$

$$\boxed{x = \frac{5\pi}{3}, \frac{4\pi}{3}}$$

$$l) \cot(x)\cos^2(x) = 2\cot(x)$$

$$\cot(x)\cos^2(x) - \cot(x) = 0 \quad \text{Res. } \sin(x) \neq 0$$

$$\cot(x)[\cos^2(x) - 2] = 0 \quad x \neq k\pi$$

$$\cot(x) = 0 \quad \cos^2(x) - 2 = 0$$

$$\cos(x) = 0 \quad \underbrace{\cos(x) = \pm\sqrt{2}}_{\text{inadmissible}}$$

$$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$$

Solving Trigonometric Equations Practice Solutions

$$m) \frac{1 + \sin(x)}{\cos(x)} + \frac{\cos(x)}{1 + \sin(x)} = 4$$

$$\frac{[1 + \sin(x)]^2 + \cos^2(x)}{\cos(x)[1 + \sin(x)]} = 4$$

Res : $\cos(x) \neq 0$ and $\sin(x) \neq -1$

$$\frac{[1 + \sin(x)]^2 + \cos^2(x)}{\cos(x)[1 + \sin(x)]} = 4$$

$$\frac{1 + 2\sin(x) + \underbrace{\sin^2(x) + \cos^2(x)}_1}{\cos(x)[1 + \sin(x)]} = 4$$

$$\frac{2\sin(x) + 2}{\cos(x)[1 + \sin(x)]} = 4$$

$$\frac{2[\cancel{\sin(x)} + 1]}{\cos(x)[\cancel{1 + \sin(x)}]} = 4$$

$$\cos(x) = \frac{1}{2}$$

$$x = 2k\pi \pm \frac{\pi}{3}$$

$$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$$

$$n) 2\sin^2(x) + 3\cos(x) - 3 = 0$$

$$2[1 - \cos^2(x)] + 3\cos(x) - 3 = 0$$

$$2\cos^2(x) - 3\cos(x) + 1 = 0$$

$$(2\cos(x) - 1)(\cos(x) - 1) = 0$$

$$\cos(x) = \frac{1}{2}, \quad \cos(x) = 1$$

$$\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}, \quad \boxed{x = 0, 2\pi}$$

Solving Trigonometric Equations Practice Solutions

o) $2\sin(x)\tan(x) - \tan(x) - 2\sin(x) + 1 = 0$

$$[2\sin(x)\tan(x) - \tan(x)] - [2\sin(x) - 1] = 0$$

$$\tan(x)[2\sin(x) - 1] - [2\sin(x) - 1] = 0$$

$$[2\sin(x) - 1][\tan(x) - 1] = 0$$

$$\sin(x) = \frac{1}{2} \quad \text{or} \quad \tan(x) = 1$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{4}, \frac{5\pi}{4}$$

p) $\cos(x)\tan(x) - 1 + \tan(x) - \cos(x) = 0$

$$[\cos(x)\tan(x) + \tan(x)] - [1 + \cos(x)] = 0$$

$$\tan(x)[\cos(x) + 1] - [1 + \cos(x)] = 0$$

$$[\cos(x) + 1][\tan(x) - 1] = 0$$

$$\cos(x) = -1 \quad \text{or} \quad \tan(x) = 1$$

$$x = \pi \quad \text{or} \quad x = \frac{\pi}{4}, \frac{3\pi}{4}$$

2. A weight hanging from a spring is set in motion moving up and down. Its distance, d , (in cm) above or below its "rest" position is described by $d(t) = 5(\sin(6t) - 4\cos(6t))$. At what times during the first 2 seconds is the weight at the rest position ($d=0$).

$$0 = 5(\sin(6t) - 4\cos(6t))$$

$$0 = \sin(6t) - 4\cos(6t)$$

$$4\cos(6t) = \sin(6t)$$

$$4 = \frac{\sin(6t)}{\cos(6t)}$$

$$4 = \tan(6t)$$

$$6t = \tan^{-1}(4)$$

$$6t = k\pi + 1.33$$

$$t = \frac{k\pi}{6} + 0.22$$

$$t = 0.22s, 0.74s, 1.27s, 1.79s$$

Solving Trigonometric Equations Practice Solutions

3. The diagram below shows the boundary of the cross section of a water channel. The

equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both

measured in cm. Find the width of the water surface when the water depth in the channel is 10 cm. [Round your answer to 2dp].

10 cm water depth corresponds to $-6 = 16\sec\left(\frac{\pi x}{36}\right) - 32$

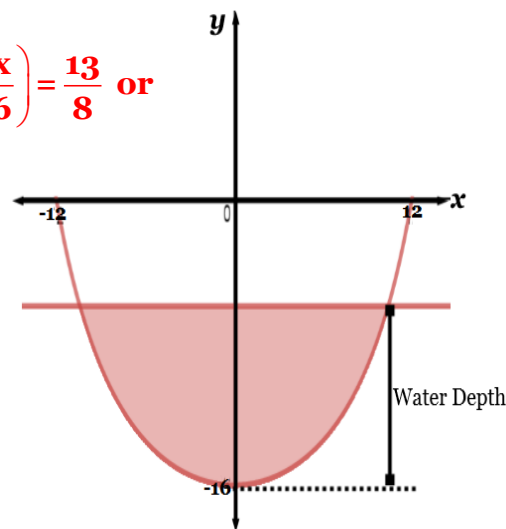
Rearranging to obtain an equation of the form $\sec\left(\frac{\pi x}{36}\right) = \frac{13}{8}$ or

$$\cos\left(\frac{\pi x}{36}\right) = \frac{8}{13}$$

$$\frac{\pi x}{36} = \cos^{-1}\left(\frac{8}{13}\right)$$

$$x = \frac{36}{\pi} \times \cos^{-1}\left(\frac{8}{13}\right)$$

$x \doteq 10.40$



4. Solve $\frac{\sin^2(x) - 3\sin(x)}{1 - 2\cos(x)} \geq 0$ over the interval $0 < x < 2\pi$.

$$\frac{\sin(x)(\sin(x) - 3)}{1 - 2\cos(x)} \geq 0$$

Zeros of numerator :

$$\sin(x) = 0$$

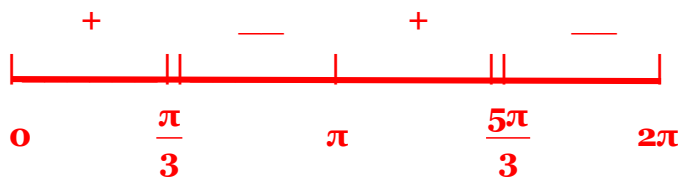
$$x = 0, \pi, 2\pi$$

Zeros of denominator :

$$1 - 2\cos(x) = 0$$

$$\cos(x) = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

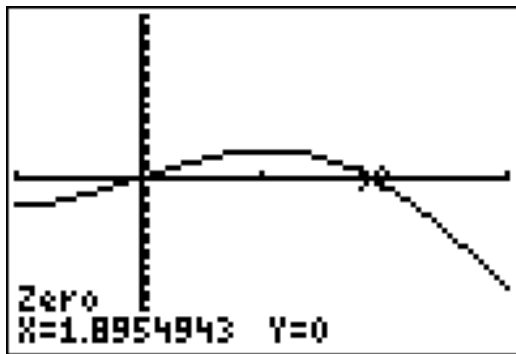


$$\therefore x \in [0, \frac{\pi}{3}) \cup [\pi, \frac{5\pi}{3})$$

Solving Trigonometric Equations Practice Solutions

5. Solve $\sin x = \frac{x}{2}$, $0 \leq x \leq 2.5$, $x \in \mathbb{R}$ (Use GDC).

$x=0$ and $x=1.8954943$



CP 11

← requires compound angles (optional question)

3*. Write $2\sin(2x) + \sqrt{12}\cos(2x)$ in the form $y = A\cos(2x - \theta)$ by finding $A > 0$ and $\theta \in [0, 2\pi]$.

$$\begin{aligned} & 2\sin(2x) + \sqrt{12}\cos(2x) \\ &= 2\sin(2x) + 2\sqrt{3}\cos(2x) \\ &= 2\sqrt{3}\cos(2x) + 2\sin(2x) \\ &= 4 \left[\underbrace{\frac{\sqrt{3}}{2}\cos(2x)}_{\cos\theta} + \underbrace{\frac{1}{2}\sin(2x)}_{\sin\theta} \right] \end{aligned}$$

$$\begin{aligned} y &= A\cos(2x - \theta) \quad \text{using identities} \\ y &= A[\cos(2x)\cos(\theta) + \sin(2x)\sin\theta] \end{aligned}$$

← comparing the expressions, $A = 4$
 $\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$$\text{If } \cos\theta = \frac{\sqrt{3}}{2} \text{ and } \sin\theta = \frac{1}{2} \quad \therefore \theta = \frac{\pi}{6}$$

$\therefore$ Expression becomes :

$$4 \left[\cos\left(\frac{\pi}{6}\right)\cos(2x) + \sin\left(\frac{\pi}{6}\right)\sin(2x) \right]$$

$$\therefore = 4\cos\left(2x - \frac{\pi}{6}\right)$$

5-3 Warm Up

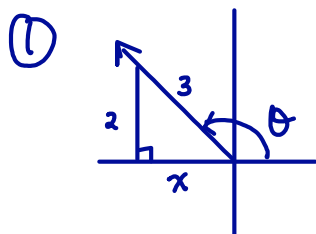
① If $\sin \theta = \frac{2}{3}$, $\frac{\pi}{2} \leq \theta \leq \pi$, find the exact value of $\sin 4\theta$.

② Find the exact value of $\cos \frac{\pi}{16}$.

③ Prove: $\frac{\cos x - \sin x}{\cos x + \sin x} = \sec 2x - \tan 2x$

④ Solve: $\sin x = \frac{x}{2}$, $x \in [0, 2.5]$

Solutions:



$$x = \sqrt{3^2 - 2^2} = \sqrt{5}$$

$$\therefore \cos \theta = -\frac{\sqrt{5}}{3}$$

$$\begin{aligned} \sin 4\theta &= 2 \sin 2\theta \cos 2\theta \\ &= 2(2 \sin \theta \cos \theta)(1 - 2 \sin^2 \theta) \\ &= 4 \left(\frac{2}{3}\right) \left(-\frac{\sqrt{5}}{3}\right) \left[1 - 2\left(\frac{2}{3}\right)^2\right] \\ &= \frac{-8\sqrt{5}}{9} \left(1 - \frac{8}{9}\right) \\ &= \frac{-8\sqrt{5}}{81} \end{aligned}$$

② $\cos 4\left(\frac{\pi}{16}\right) = \cos \frac{\pi}{4}$

$$2 \cos^2 \left[2\left(\frac{\pi}{16}\right)\right] - 1 = \cos \frac{\pi}{4}$$

$$\cos^2 \left[2\left(\frac{\pi}{16}\right)\right] = \frac{\cos \frac{\pi}{4} + 1}{2}$$

Q1 $\rightarrow$ + only

$$\cos \left[2\left(\frac{\pi}{16}\right)\right] = \sqrt{\frac{\cos \frac{\pi}{4} + 1}{2}}$$

$$2 \cos^2 \left(\frac{\pi}{16}\right) - 1 = \sqrt{\frac{\frac{1}{\sqrt{2}} + 1}{2}}$$

Q1 $\rightarrow$ + only

$$\cos \frac{\pi}{16} = \sqrt{\frac{\sqrt{\frac{\frac{1}{\sqrt{2}} + 1}{2}} + 1}{2}}$$

$$\cos \frac{\pi}{16} = \sqrt{\frac{\sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}} + 1}{2}}$$

$$= \sqrt{\frac{1}{2} \sqrt{\frac{1+\sqrt{2}}{2\sqrt{2}}} + \frac{1}{2}}$$

$$= \sqrt{\frac{1}{2} \sqrt{\frac{\sqrt{2}+2}{4}} + \frac{1}{2}}$$

$$= \sqrt{\frac{\sqrt{\sqrt{2}+2}}{4} + \frac{1}{2}}$$

$$= \frac{\sqrt{\sqrt{\sqrt{2}+2}+2}}{2}$$

optional steps
to simplify

5-3 Warm Up sol'ns cont'd

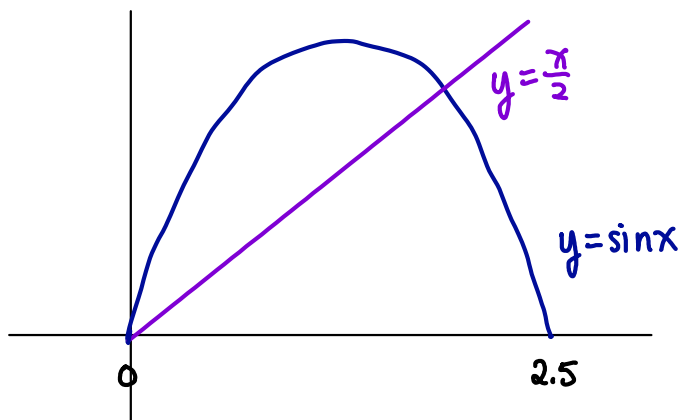
$$\begin{aligned}
 \textcircled{3} \quad RS &= \sec 2x - \tan 2x \\
 &= \frac{1}{\cos 2x} - \frac{\sin 2x}{\cos 2x} \\
 &= \frac{1 - 2 \sin x \cos x}{\cos^2 x - \sin^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x - 2 \sin x \cos x}{(\cos x + \sin x)(\cos x - \sin x)} \\
 &= \frac{(\cos x - \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)} \\
 &= \frac{\cos x - \sin x}{\cos x + \sin x} \\
 &= LS \quad \because LS = RS \\
 &\quad \therefore \text{The identity is true}
 \end{aligned}$$

Alternate solution to ②:

$$\begin{aligned}
 \cos 4\left(\frac{\pi}{16}\right) &= 2 \cos^2\left[2\left(\frac{\pi}{16}\right)\right] - 1 \\
 \frac{\frac{\sqrt{2}}{2} + 1}{2} &= \cos^2\left[2\left(\frac{\pi}{16}\right)\right] \\
 \sqrt{\frac{\sqrt{2} + 2}{4}} &= \cos\left[2\left(\frac{\pi}{16}\right)\right] \\
 \frac{\sqrt{\sqrt{2} + 2}}{2} &= 2 \cos^2\left(\frac{\pi}{16}\right) - 1 \\
 \frac{\frac{\sqrt{\sqrt{2} + 2}}{2} + 1}{2} &= \cos^2\left(\frac{\pi}{16}\right) \\
 \frac{\sqrt{\sqrt{2} + 2} + 2}{4} &= \cos^2\left(\frac{\pi}{16}\right) \\
 \frac{\sqrt{\sqrt{\sqrt{2} + 2} + 2}}{2} &= \cos\left(\frac{\pi}{16}\right)
 \end{aligned}$$

④ Graph $y_1 = \sin x$ and $y_2 = \frac{x}{2}$

$x \in [0, 2.5]$



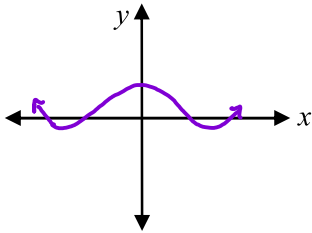
$x_1 \approx 1.8954943$

$x_2 = 0$

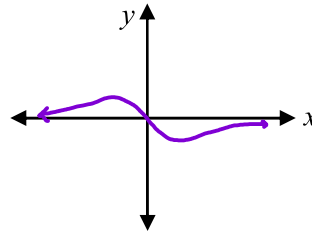
Derivatives of the Sine, Cosine and Tangent Functions

Using the GDC, determine the following:

a) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$



b) $\lim_{x \rightarrow 0} \frac{\cos x - 1}{x} = 0$



Using the above limits, the identity: $\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b)$, and first principles, determine the derivative of $y = \sin(x)$.

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x) \cos(h) + \cos(x) \sin(h) - \sin(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x) [\cos(h) - 1] + \sin(h) \cos(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[\frac{\sin(x) [\cos(h) - 1]}{h} + \frac{\sin(h) \cos(x)}{h} \right]$$

$$= \lim_{h \rightarrow 0} \frac{\sin(x) [\cos(h) - 1]}{h} + \lim_{h \rightarrow 0} \frac{\sin(h) \cos(x)}{h}$$

$$= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}$$

$$= \sin(x)(0) + \cos(x)(1)$$

$$= \cos(x)$$

common factor $\sin(x)$ from the first and last terms of the numerator

Limit of the sum is the sum of the limits.

Note: $\cos(x)$ and $\sin(x)$ are not dependent on h
 $\therefore$ they can be written as factors to the left of the limit.

Therefore, the derivative of $y = \sin(x)$ is $y' = \cos(x)$ (or $\frac{d}{dx} \sin(x) = \cos(x)$).

In general, if $y = [\sin(g(x))]^n$, then $y' = n \cdot [\sin(g(x))]^{n-1} \cdot \cos[g(x)] \cdot g'(x)$.

Using the previous limits, the identity: $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, and first principles, determine the derivative of $y = \cos(x)$.

$$\begin{aligned}
 \frac{dy}{dx} &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x)[\cos(h) - 1] - \sin(x)\sin(h)}{h} \\
 &= \cos(x) \lim_{h \rightarrow 0} \frac{\cos(h) - 1}{h} - \sin(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
 &= \cos(x)(0) - \sin(x)(1) \\
 &= -\sin(x)
 \end{aligned}$$

Therefore, the derivative of $y = \cos(x)$ is $-\sin(x)$ (or $\frac{d}{dx} \cos(x) = -\sin(x)$).

In general, if $y = [\cos(g(x))]^n$, then $y' = \underline{n \cdot [\cos(g(x))]^{n-1} \cdot -\sin(g(x)) \cdot g'(x)}$.

Determine the derivative of $y = \tan(x)$.

$$\begin{aligned}
 y &= \frac{\sin(x)}{\cos(x)} \\
 y' &= \frac{\cos(x)(\cos(x)) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
 &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
 &= \frac{1}{\cos^2(x)} \\
 &= \sec^2(x)
 \end{aligned}$$

Therefore, the derivative of $y = \tan(x)$ is $\sec^2(x)$ (or $\frac{d}{dx} \tan(x) = \sec^2(x)$).

In general, if $y = [\tan(g(x))]^n$, then $y' = \underline{n \cdot [\tan(g(x))]^{n-1} \cdot \sec^2(g(x)) \cdot g'(x)}$.

Determine the derivative of the following:

a) $y = \csc(x)$

$$y = \frac{1}{\sin(x)}$$

$$y' = \frac{0(\sin(x)) - 1(\cos(x))}{(\sin(x))^2}$$

$$y' = \frac{-\cos(x)}{\sin^2(x)}$$

$$y' = -\csc(x)\cot(x)$$

$$y = (\sin(x))^{-1}$$

$$y' = -(\sin(x))^{-2} \cdot \cos(x)$$

$$= \frac{-\cos(x)}{\sin^2(x)}$$

$$= -\frac{1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)}$$

$$= -\csc(x)\cot(x)$$

b) $y = \sec(x)$

$$y = \frac{1}{\cos(x)}$$

$$y' = \frac{0(\cos(x)) - 1(-\sin(x))}{(\cos(x))^2}$$

$$y' = \frac{\sin(x)}{\cos^2(x)}$$

$$y' = \sec(x)\tan(x)$$

$$y = (\cos(x))^{-1}$$

$$y' = -(\cos(x))^{-2}(-\sin(x))$$

$$= \frac{\sin(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)}$$

$$= \sec(x)\tan(x)$$

c) $y = \cot(x)$

$$y = \frac{\cos(x)}{\sin(x)}$$

$$y' = \frac{-\sin(x)[\sin(x)] - \cos(x)[\cos(x)]}{[\sin(x)]^2}$$

$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)}$$

$$= \frac{-[\sin^2(x) + \cos^2(x)]}{\sin^2(x)}$$

$$= \frac{-1}{\sin^2(x)}$$

$$= -\csc^2(x)$$

or $y = [\tan(x)]^{-1}$

$$y' = -1[\tan(x)]^{-2} \cdot \sec^2(x)$$

$$= \frac{\sec^2(x)}{\tan^2(x)}$$

$$= \frac{1}{\cos^2(x)} \cdot \frac{\cos^2(x)}{\sin^2(x)}$$

$$= \frac{1}{\sin^2(x)}$$

$$= -\csc^2(x)$$

Derivatives of Secondary

Trigonometric Ratios

In general,

$$\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx} \sec(x) = \sec(x)\tan(x)$$

$$\frac{d}{dx} \cot(x) = -\csc^2(x)$$

Example 1: Differentiate the following functions:

a. $y = \sin^2(x)$

$$y' = 2 \sin(x) \cos(x)$$

b. $y = \sin(x^2)$

$$y' = \cos(x^2) \cdot (2x) = 2x \cos(x^2)$$

c. $y = \sec(x)$

$$y = \frac{1}{\cos(x)}$$

$$y' = \frac{0 \cdot \cos(x) - 1[-\sin(x)]}{[\cos(x)]^2}$$

$$y' = \frac{\sin(x)}{\cos^2(x)} \Rightarrow y' = \sec(x) \tan(x)$$

d. $y = (\tan(3x))\sqrt{\cos(x)}$

$$y' = \sec^2(3x) \cdot 3 \cdot \sqrt{\cos(x)} + [\tan(3x)] \cdot \frac{1}{2} [\cos(x)]^{-\frac{1}{2}} \cdot [-\sin(x)]$$

$$y' = 3 \sec^2(3x) \cdot \sqrt{\cos(x)} - \frac{\sin(x) \cdot \tan(3x)}{2 \sqrt{\cos(x)}}$$

or $y = [\cos(x)]^{-1}$

$$y' = -1 [\cos(x)]^{-2} [-\sin(x)]$$

$$y' = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} = \sec(x) \tan(x)$$

Example 2: Find the equation of the line that is tangent to the curve $y = \frac{\cos(x)}{x}$ at the point on the curve where $x = \pi$

1) Derivative

$$\frac{dy}{dx} = \frac{\sin(x) \cdot x - \cos(x)}{x^2}$$

2) Evaluate $\frac{dy}{dx} \Big|_{x=\pi}$

$$\frac{dy}{dx} \Big|_{x=\pi} = \frac{\pi \sin(\pi) - \cos(\pi)}{(\pi)^2}$$

$$= \frac{0 - (-1)}{\pi^2}$$

$$= \frac{1}{\pi^2}$$

3) Tangent point at $x = \pi$:

$$y = \frac{\cos(\pi)}{\pi}$$

$$= -\frac{1}{\pi}$$

4) Equation

$$y - (-\frac{1}{\pi}) = (\frac{1}{\pi^2})[x - \pi]$$

$$y = \frac{1}{\pi^2}x - \frac{2}{\pi}$$

$\therefore$ The equation of the tangent line at $x = \pi$ is $y = \frac{1}{\pi^2}x - \frac{2}{\pi}$

Example 3: Find all extrema for $y = 2x - \tan(x)$, $x \in [0, \pi]$

$$y' = 2 - \sec^2(x)$$

$$= 2 - \frac{1}{\cos^2(x)}$$

critical #s $\Rightarrow y' = 0$:

$$\frac{2\cos^2(x) - 1}{\cos^2(x)} = 0$$

$$\cos^2(x) = \frac{1}{2}$$

$$\cos(x) = \pm \frac{1}{\sqrt{2}}$$

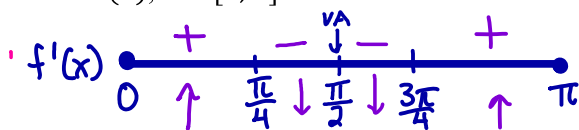
$$x = \frac{\pi}{4}$$

$\therefore \cos$ is +ve and -ve, consider all quadrants

$$\therefore x_1 = \frac{\pi}{4}, x_2 = \frac{3\pi}{4}$$

$$y' \geq 0 \Rightarrow \cos^2(x) = 0$$

$$x = \frac{\pi}{2}$$



At $x = \frac{\pi}{4}$, $y = 2(\frac{\pi}{4}) - \tan(\frac{\pi}{4})$

$$= \frac{\pi}{2} - 1$$

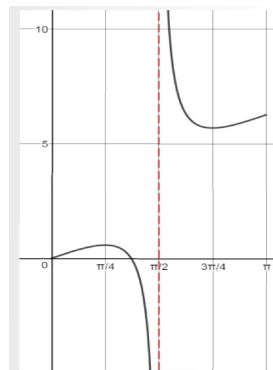
$$= \frac{\pi - 2}{2}$$

At $x = \frac{3\pi}{4}$, $y = 2(\frac{3\pi}{4}) - \tan(\frac{3\pi}{4})$

$$= \frac{3\pi}{2} + 1$$

$$= \frac{3\pi + 2}{2}$$

$\therefore$ Local max at $(\frac{\pi}{4}, \frac{\pi-2}{2})$, local min at $(\frac{3\pi}{4}, \frac{3\pi+2}{2})$



Example 4: Find the derivative: $y = \tan(\sin(\cos(x^2)))$

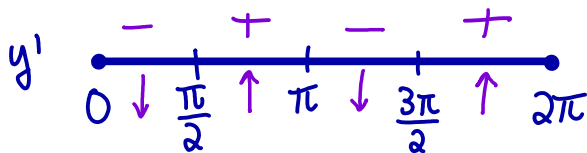
$$\begin{aligned} y' &= \sec^2[\sin[\cos(x^2)]] \cdot \cos[\cos(x^2)] \cdot -\sin(x^2) \cdot 2x \\ &= -2x \sin(x^2) \cdot \cos[\cos(x^2)] \cdot \sec^2[\sin[\cos(x^2)]] \end{aligned}$$

Example 5: Find all local extrema for $y = 2\cos^2(x) + 1$, $x \in [0, 2\pi]$

$$\begin{aligned} y' &= 2[2\cos(x) \cdot -\sin(x)] \\ &= -4\sin(x)\cos(x) \end{aligned}$$

$$\begin{aligned} \text{critical pts} &\Rightarrow y' = 0 \\ 0 &= -4\sin(x)\cos(x) \end{aligned}$$

$$\begin{aligned} \sin(x) = 0 \quad \text{or} \quad \cos(x) = 0 \\ x_1 = 0 \quad \quad \quad x_4 = \frac{\pi}{2} \\ x_2 = \pi \quad \quad \quad x_5 = \frac{3\pi}{2} \\ x_3 = 2\pi \end{aligned}$$



$\therefore$ Local min points occur at $(\frac{\pi}{2}, 1), (\frac{3\pi}{2}, 1)$
and local max point at $(\pi, 3)$.

$$\begin{aligned} \text{At } x=0, \quad y &= 2\cos^2(0) + 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

$$\begin{aligned} \text{At } x=\pi, \quad y &= 2\cos^2(\pi) + 1 \\ &= 2 + 1 \\ &= 3 \end{aligned}$$

$$\text{At } x=2\pi, \quad y = 2\cos^2(2\pi) + 1$$

$$\begin{aligned} \text{At } x = \frac{\pi}{2}, \quad y &= 2\cos^2\left(\frac{\pi}{2}\right) + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{At } x = \frac{3\pi}{2}, \quad y &= 2\cos^2\left(\frac{3\pi}{2}\right) + 1 \\ &= 1 \end{aligned}$$

Practice Questions 5.2: **Derivatives of Trigonometric Functions-Solutions**

1. Multiple Choice. Identify the choice that best completes the statement or answers the question.

A 1. If $f(x) = 2\cos(3x)$, find $f'(\frac{\pi}{3})$.

- a. 0 b. 3 c. 6 d. -6

$$\begin{aligned} f(x) &= 2\cos(3x) \\ f'(x) &= -2(3)\sin(3x) \\ f'\left(\frac{\pi}{3}\right) &= -6\sin\left(3\left(\frac{\pi}{3}\right)\right) \\ &= 0 \end{aligned}$$

C 2. If $f(x) = \sin x + \cos x + x$, find $f'(\pi)$.

- a. 2 b. π c. 0 d. $1+\pi$

$$\begin{aligned} f(x) &= \sin x + \cos x + x \\ f'(x) &= \cos x - \sin x + 1 \\ f'(\pi) &= -1 + 0 + 1 \\ &= 0 \end{aligned}$$

D 3. If $y = \tan 2x$, find y'' at $x = \frac{\pi}{2}$.

- a. $\frac{\sqrt{3}}{8}$ b. $8\sqrt{3}$
c. -8 d. 0

$$\begin{aligned} f(x) &= \tan 2x \\ f'(x) &= 2\sec^2(2x) \\ f''(x) &= 8\sec^2(2x)\tan(2x) \\ f''\left(\frac{\pi}{2}\right) &= 8\sec^2(\pi)\tan(\pi) \\ &= 0 \end{aligned}$$

A 4. Find the slope of the tangent to $y = \cos^2 x$ at the point $(\frac{\pi}{3}, \frac{1}{4})$.

- a. $-\frac{\sqrt{3}}{2}$ b. 1 c. 2 d. $\frac{\sqrt{3}}{2}$

$$\begin{aligned} f(x) &= \cos^2 x \\ f'(x) &= -2\cos x \sin x = -\sin 2x \\ f'\left(\frac{\pi}{3}\right) &= -\sin\left(\frac{2\pi}{3}\right) \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

D 5. Find the slope of the tangent to $y = x^2 \cos(4x^2 + 7)$ at the point where $x=1$.

- a. 6.011 b. 3.023
c. -4.203 d. 8.009

$$\begin{aligned} f(x) &= x^2 \cos(4x^2 + 7) \\ f'(x) &= 2x \cos(4x^2 + 7) - 8x^3 \sin(4x^2 + 7) \\ f'(1) &= 2 \cos(11) - 8 \sin(11) \\ &= 8.009 \end{aligned}$$

6. Differentiate the following:

a. $f(x) = \cos(3x^2)$ $f'(x) = -6x \sin(3x^2)$	b. $y = \sin \sqrt{x^2 - 1}$ $y' = \frac{2x}{\sqrt{x^2 - 1}} \cos \sqrt{x^2 - 1}$ $= \frac{x \cos \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}$
c. $y = (2x^2 - 4x) \cos^3(2x)$ $y' = (4x - 4) \cos^3(2x) + (2x^2 - 4x) 3 \cos^2(2x) (-\sin(2x))(2)$ $= 4 \cos^2(2x) [(x - 4) \cos(2x) - 3(x^2 - 2x) \sin(2x)]$	d. $y = (1 - 3 \cos^2 x)^4$ $y' = 4(1 - 3 \cos^2 x)^3 (6 \cos x \sin x)$ $= 12 \sin 2x (1 - 3 \cos^2 x)^3$
e. $f(x) = (\sin 3x + \cos 3x)^4$ $f'(x) = 4(\sin 3x + \cos 3x)^3 (3 \cos 3x - 3 \sin 3x)$ $= 12(\sin 3x + \cos 3x)^3 (\cos 3x - \sin 3x)$	f. $y = \frac{\tan(3x)}{\sin(2x)}$ $y' = \frac{3 \sec^2 3x (\sin 2x) - \tan(3x) (2 \cos(2x))}{\sin^2(2x)}$

7. If $y = A(\cos kt) + B(\sin kt)$, where A, B, and k are constants, show that: $y'' + k^2 y = 0$.

$$\begin{aligned}
 y &= A \cos(kt) + B \sin(kt) \\
 y' &= -Ak \sin(kt) + Bk \cos(kt) \\
 y'' &= -Ak^2 \cos(kt) - Bk^2 \sin(kt) \\
 &= -k^2 (A \cos(kt) + B \sin(kt)) \\
 &= -k^2 y \\
 \therefore y'' + k^2 y &= 0
 \end{aligned}$$

8. Prove that $y = \sec x + \tan x$ is always increasing on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$\begin{aligned}
 y &= \sec x + \tan x \\
 y' &= \sec x \tan x + \sec^2 x \\
 &= \frac{\sin x + 1}{\cos^2 x} \\
 -\frac{\pi}{2} &< x < \frac{\pi}{2} \\
 -1 &< \sin x < 1 \\
 0 &< \sin x + 1 < 2 \\
 \text{since for } x &\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \sin x + 1 > 0 \text{ and } \cos^2 x > 0 \Rightarrow y' = \frac{\sin x + 1}{\cos^2 x} > 0.
 \end{aligned}$$

That means $\sec x + \tan x$ is always increasing on this interval.

5-4 Warm Up

1. Solve for x : $\sqrt{3} \cos(x) = \sin(2x)$, $-2\pi \leq x \leq 2\pi$
2. Solve: $\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) = \sqrt{2}$, $0 \leq x \leq 2\pi$
3. Prove the following identities:
 - a) $\frac{1}{\cos(x) \sin(x)} + \frac{1}{\cos(x) - \sin(x)} = \frac{2 \cos(x)}{\cos(2x)}$
 - b) $2 \cos^3(x) + \sin(2x) \sin(x) = 2 \cos(x)$
4. Differentiate with respect to x :
 - a) $y = \cos^3(x)$
 - b) $y = \sin(x)$
 - c) $y = x^3 + \sin^2(x)$
 - d) $y = \sin^4(x^3 - 1)$
5. Find the equation of the tangent line to the curve $f(x) = 6 \tan(x) - \tan(2x)$ at $x = 0$.

5-4 Warm Up Solutions

$$\textcircled{1} \quad \sqrt{3} \cos x = \sin 2x$$

$$2 \sin x \cos x - \sqrt{3} \cos x = 0$$

$$\cos x (2 \sin x - \sqrt{3}) = 0$$

$$\cos x = 0 \quad \text{or} \quad \sin x = \frac{\sqrt{3}}{2}$$

$$x_1 = \frac{\pi}{2}$$

$$x = \frac{\pi}{3}$$

$$x_2 = \frac{3\pi}{2}$$

$$\therefore \sin + \tan \text{ Q1 \& Q2}$$

$$x_3 = -\frac{\pi}{2}$$

$$\therefore x_5 = \frac{\pi}{3}$$

$$x_4 = -\frac{3\pi}{2}$$

$$x_6 = \frac{2\pi}{3}$$

$$x_7 = \frac{\pi}{3} - 2\pi$$

$$= -\frac{5\pi}{3}$$

$$x_8 = \frac{2\pi}{3} - 2\pi$$

$$= -\frac{4\pi}{3}$$

$$\textcircled{2} \quad \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right) = \sqrt{2} \quad \text{square both sides}$$

$$\sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) = 2$$

$$1 + \sin 2\left(\frac{x}{2}\right) = 2$$

$$\sin x = 1$$

$$\therefore x = \frac{\pi}{2}$$

$$\textcircled{3} \text{ a) } \frac{1}{\cos x + \sin x} + \frac{1}{\cos x - \sin x} = \frac{2 \cos x}{\cos 2x}$$

$$\text{LS} = \frac{1}{\cos x + \sin x} + \frac{1}{\cos x - \sin x}$$

$$= \frac{\cos x - \sin x + \cos x + \sin x}{\cos^2 x - \sin^2 x}$$

$$= \frac{2 \cos x}{\cos 2x}$$

$$= \text{RS}$$

$$\text{b) } 2 \cos^3 x + \sin 2x \sin x = 2 \cos x$$

$$\text{LS} = 2 \cos^3 x + \sin 2x \sin x$$

$$= 2 \cos^3 x + 2 \sin^2 x \cos x$$

$$= 2 \cos x (\cos^2 x + \sin^2 x)$$

$$= 2 \cos x$$

$$= \text{RS}$$

$$\textcircled{4} \quad \text{a) } y = \cos^3 x$$

$$y' = 3\cos^2 x (-\sin x) \\ = -3\sin x \cos^2 x$$

$$\text{b) } y = \sin 3x$$

$$y' = 3\cos 3x$$

$$\text{c) } y = x^3 + \sin^2 x$$

$$y' = 3x^2 + 2\sin x \cos x$$

$$\text{d) } y = \sin^4(x^3 - 1)$$

$$y' = 4\sin^3(x^3 - 1) \cdot \cos(x^3 - 1) \cdot 3x^2$$

$$\textcircled{5} \quad f'(x) = 6\sec^2 x - \sec^2(2x) \cdot 2 \\ = 6\sec^2 x - 2\sec^2(2x)$$

$$f'(0) = 6\sec^2(0) - 2\sec^2[2(0)] \\ = 6(1)^2 - 2(1)^2 \\ = 4$$

$$f(0) = 6\tan(0) - \tan[2(0)] \\ = 6(0) - 0 \\ = 0$$

$$\therefore y - (0) = 4[x - (0)] \\ y = 4x$$

$\therefore$ The equation of the tangent at $x=0$ is $y=4x$.

Warm-up: Derivatives of $\sin x$, $\cos x$, and $\tan x$

1. Find $\frac{dy}{dx}$ for each of these functions.

a) $y = 2\sqrt{\cos 4x}$

$$\begin{aligned}\frac{dy}{dx} &= 2 \left[\frac{1}{2} (\cos 4x)^{-\frac{1}{2}} \cdot (-\sin 4x) \cdot 4 \right] \\ &= \frac{-4 \sin 4x}{\sqrt{\cos 4x}}\end{aligned}$$

c) $y = \frac{3}{5} \tan(x^5)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{3}{5} [\sec^2(x^5)] (5x^4) \\ &= 3x^4 \sec^2(x^5)\end{aligned}$$

e) $y = \frac{5}{4-3\tan x}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{-5(-3\sec^2 x)}{(4-3\tan x)^2} \\ \frac{dy}{dx} &= \frac{15\sec^2 x}{(4-3\tan x)^2}\end{aligned}$$

b) $y = (1 + \sin \sqrt{x})^3$

$$\begin{aligned}\frac{dy}{dx} &= 3(1 + \sin \sqrt{x})^2 [\cos \sqrt{x} \cdot (\frac{1}{2}x^{-\frac{1}{2}})] \\ &= \frac{3\cos \sqrt{x}(1 + \sin \sqrt{x})^2}{2\sqrt{x}}\end{aligned}$$

d) $y = 2\sin^2 x \cos 3x$

$$\begin{aligned}\frac{dy}{dx} &= 4\sin x \cos x \cos 3x + 2\sin^2 x (-\sin 3x)(3) \\ &= 4\sin x \cos x \cos 3x - 6\sin 3x \sin^2 x \\ &= 2\sin x (2\cos x \cos 3x - 3\sin x \sin 3x)\end{aligned}$$

f) $x + y \sin x^2 = 1 - \cos^3 y$

$$\begin{aligned}1 + \frac{dy}{dx} \sin x^2 + y(\cos x^2)(2x) &= -3\cos^2 y \cdot (-\sin y) \frac{dy}{dx} \\ 1 + \sin x^2 \frac{dy}{dx} + 2xy \cos x^2 &= 3\sin y \cos^2 y \frac{dy}{dx} \\ \frac{dy}{dx} (\sin x^2 - 3\sin y \cos^2 y) &= -1 - 2xy \cos x^2 \\ \frac{dy}{dx} &= \frac{-1 - 2xy \cos x^2}{(\sin x^2 - 3\sin y \cos^2 y)}\end{aligned}$$

2. Find the equation of the normal to the curve $y = x + \tan x$ at the point where $x = \frac{\pi}{4}$.

$$m = \frac{dy}{dx} = 1 + \sec^2 x$$

$$\left. \frac{dy}{dx} \right|_{x=\frac{\pi}{4}} = 1 + \sec^2\left(\frac{\pi}{4}\right)$$

$$= 1 + \frac{1}{(\cos \frac{\pi}{4})^2}$$

$$= 1 + \frac{1}{(\frac{1}{\sqrt{2}})^2}$$

$$= 1 + 2$$

$$= 3$$

$$m_{\text{normal}} = -\frac{1}{3}$$

$$\text{At } x = \frac{\pi}{4}, y = \frac{\pi}{4} + \tan \frac{\pi}{4}$$

$$y = \frac{\pi}{4} + 1$$

$\therefore$ The equation of the normal is:

$$y - \left(\frac{\pi}{4} + 1\right) = -\frac{1}{3} \left(x - \frac{\pi}{4}\right)$$

$$y = -\frac{1}{3}x + \frac{\pi}{12} + \frac{\pi}{4} + 1$$

$$12y = -4x + \pi + 3\pi + 12$$

$$4x + 12y - 4\pi - 12 = 0$$

$$x + 3y - \pi - 3 = 0$$

Note: IB: $ax + by = c$ form

$$x + 3y = \pi + 3$$

Differentiation Rules for Trigonometric Functions

Recall:

- The Product Rule: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
- The Chain Rule: $\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \frac{d}{dx}[g(x)]$
- The Quotient Rule: $\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$

Ex. 1. Differentiate with respect to x .

a) $y = \sin(2x^3)$

$$\begin{aligned} y' &= [\cos(2x^3)](6x^2) \\ &= 6x^2 \cos(2x^3) \end{aligned}$$

b) $f(x) = \frac{-1}{4} \csc(-8x)$

$$\begin{aligned} f'(x) &= -\frac{1}{4} [-\csc(-8x) \cot(-8x)](-8) \\ &= -2 \csc(-8x) \cot(-8x) \end{aligned}$$

c) $y = \cos^3(x^2 - 1)$

$$\begin{aligned} y' &= 3 \cos^2(x^2 - 1) \cdot [-\sin(x^2 - 1)] \cdot 2x \\ &= -6x \sin(x^2 - 1) \cdot \cos^2(x^2 - 1) \end{aligned}$$

d) $y = \cot^3(1 + x^2)$

$$\begin{aligned} y' &= 3 \cot^2(1 + x^2) \cdot [-\csc^2(1 + x^2)] \cdot 2x \\ &= -6x \cot^2(1 + x^2) \cdot \csc^2(1 + x^2) \end{aligned}$$

Ex. 2. Find the equation of the tangent line to $y = \frac{\sin x}{\cos x}$ at $x = \frac{\pi}{3}$.

$$y = \tan x$$

$$y' = \sec^2 x$$

$$\text{At } x = \frac{\pi}{3}, y' = \sec^2 \frac{\pi}{3}$$

$$y' = \frac{1}{\cos^2 \frac{\pi}{3}}$$

$$y' = \frac{1}{\left(\frac{1}{2}\right)^2}$$

$$y' = 4$$

$$\text{At } x = \frac{\pi}{3}, y = \tan \frac{\pi}{3}$$

$$y = \sqrt{3}$$

$\therefore$ The equation of the tangent line is:

$$y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$$

$$y = 4x - \frac{4\pi}{3} + \sqrt{3}$$

$$3y = 12x - 4\pi + 3\sqrt{3}$$

$$12x - 3y - 4\pi + 3\sqrt{3} = 0 \quad (\text{standard form})$$

Application of derivatives of trigonometric functions

A point on a curve at which the gradient is zero, where $\frac{dy}{dx} = 0$, is called a stationary point.

Three types: Minimum point, Maximum point, Point of inflexion

1. Find and classify the stationary points on the curve $y = x + \sin x$ in the range $0 \leq x \leq 2\pi$. Sketch the curve using the GDC.

2. Given that $y = \sin x(1 - \cos x)$, show that $\frac{dy}{dx} = (1 + 2\cos x)(1 - \cos x)$

Hence, find the coordinates of the points on the curve $y = \sin x(1 - \cos x)$, in the range $0 \leq x \leq \pi$, where the gradient is zero.

1) $f(x) = x + \sin x$

$f'(x) = 1 + \cos x$

Critical pts $\Rightarrow f'(x) = 0$

$1 + \cos x = 0$

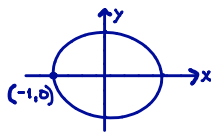
$\cos x = -1$

$x = \pi$

$f(\pi) = \pi + \sin \pi$

$y = \pi$

(π, π) is a critical point.



	$(0, \pi)$	$(\pi, 2\pi)$
$1 + \cos x$	+	+
$f'(x)$	+	+
$f(x)$	$\uparrow$	$\uparrow$

No max/min point

$f''(x) = -\sin x$

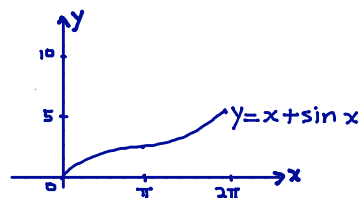
Possible P.O.I $\Rightarrow f''(x) = 0$

$-\sin x = 0$

$x = 0, \pi$

	$(0, \pi)$	$(\pi, 2\pi)$
$-\sin x$	-	+
$f''(x)$	-	+
$f(x)$	$\cup$	$\cap$

$\therefore$ There is a point of inflection at (π, π) .



2) $y = \sin x(1 - \cos x)$

$= \sin x - \sin x \cos x$

$\frac{dy}{dx} = \cos x - [\cos x \cos x + \sin x(-\sin x)]$

$= \cos x - \cos^2 x + \sin^2 x$

$= \cos x - \cos^2 x + 1 - \cos^2 x$

$= 1 + \cos x - 2\cos^2 x$

$= (1 + 2\cos x)(1 - \cos x)$

$\therefore \frac{dy}{dx} = 0$

$\therefore 1 + 2\cos x = 0$ or $1 - \cos x = 0$

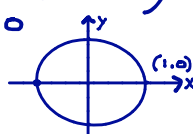
$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}$

$\cos x = 1$

$x = 0$

$\therefore x = \frac{2\pi}{3}, 0 \leq x \leq \pi$



At $x = 0, y = 0$

At $x = \frac{2\pi}{3}$

$y = \sin \frac{2\pi}{3} (1 - \cos \frac{2\pi}{3})$

$= \frac{\sqrt{3}}{2} (1 - (-\frac{1}{2}))$

$= \frac{\sqrt{3}}{2} (\frac{3}{2})$

$= \frac{3\sqrt{3}}{4}$

$\therefore$ The points are $(0, 0)$ and $(\frac{2\pi}{3}, \frac{3\sqrt{3}}{4})$

Optimization Problems with Trigonometric Functions

Ex 1: A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side through an angle, θ . How should θ be chosen so that the gutter will carry the maximum amount of water?

Let the area of the cross-section be A .

$$\text{Area of trapezoid} = \frac{(a+b)h}{2}$$

$$\begin{aligned} A &= \frac{1}{2}(a+b)(h) \\ &= \frac{1}{2}[10 + (10 + 20 \cos(\theta))] [10 \sin(\theta)] \\ &= 100 \sin(\theta) [\cos(\theta) + 1] \end{aligned}$$

For maximum, find $A' = 0$:

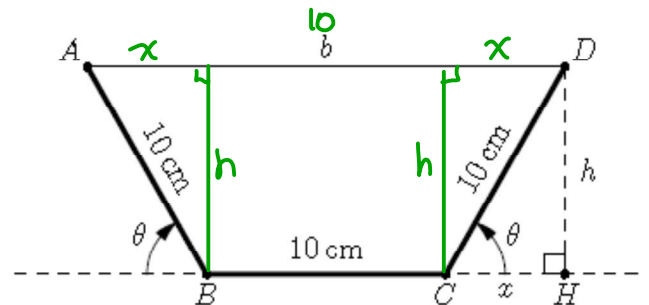
$$\begin{aligned} A' &= 100 \cos(\theta) [\cos(\theta) + 1] + 100 \sin(\theta) [-\sin(\theta)] \\ &= 100 \cos^2(\theta) + 100 \cos(\theta) - 100 \sin^2(\theta) \\ &= 100 \cos^2(\theta) + 100 \cos(\theta) - 100 [1 - \cos^2(\theta)] \\ &= 200 \cos^2(\theta) + 100 \cos(\theta) - 100 \\ &= 100 [2 \cos(\theta) - 1] [\cos(\theta) + 1] \end{aligned}$$

critical #s $\Rightarrow A' = 0$,

$$\begin{aligned} 2 \cos(\theta) - 1 &= 0 \\ \cos(\theta) &= \frac{1}{2} \\ \theta &= \frac{\pi}{3} \\ \therefore 0 < \theta < \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} \text{or } \cos(\theta) + 1 &= 0 \\ \cos(\theta) &= -1 \\ \theta &= \pi \\ \text{inadmissible } \therefore 0 < \theta < \frac{\pi}{2} \end{aligned}$$

$\therefore \theta$ should be at $\frac{\pi}{3}$ radians.



$$\sin(\theta) = \frac{h}{10}$$

$$10 \sin(\theta) = h$$

$$\cos(\theta) = \frac{x}{10}$$

$$10 \cos(\theta) = x$$

Note: the second derivative test can be used to verify that $\theta = \frac{\pi}{3}$ results in the max area.

Ex 2: A ladder is to be carried down a hallway 2 m wide. Unfortunately at the end of the hallway there is a right-angled turn into a hallway 3 m wide. What is the length of the longest ladder that can be carried **horizontally** around the corner?

The goal is shifted from finding the longest ladder that will go around the corner to finding the shortest ladder that will get stuck.

Let the ladder be L , where $L = l_1 + l_2$, and θ be the angle the ladder meets the corner, as shown. Note: $0 < \theta < \frac{\pi}{2}$

$$\sin(\theta) = \frac{3}{l_2} \quad \cos(\theta) = \frac{2}{l_1}$$

$$l_2 = \frac{3}{\sin(\theta)} \quad l_1 = \frac{2}{\cos(\theta)}$$

$$\begin{aligned} L &= l_1 + l_2 \\ &= \frac{3}{\sin(\theta)} + \frac{2}{\cos(\theta)} \\ &= 3[\sin(\theta)]^{-1} + 2[\cos(\theta)]^{-1} \end{aligned}$$

$$\begin{aligned} L' &= -3[\sin(\theta)]^{-2}[\cos(\theta)] - 2[\cos(\theta)]^{-2}[-\sin(\theta)] \\ &= \frac{-3\cos(\theta)}{\sin^2(\theta)} + \frac{2\sin(\theta)}{\cos^2(\theta)} \end{aligned}$$

critical #s $\Rightarrow L' = 0$

$$\frac{-3\cos^3(\theta) + 2\sin^3(\theta)}{\sin^2(\theta)\cos^2(\theta)} = 0$$

$$2\sin^3(\theta) = 3\cos^3(\theta)$$

$$\tan^3(\theta) = \frac{3}{2}$$

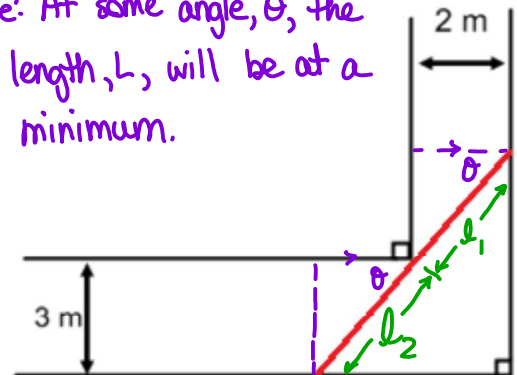
$$\tan(\theta) = \left(\frac{3}{2}\right)^{\frac{1}{3}}$$

$$\theta = \tan^{-1}\left(\left(\frac{3}{2}\right)^{\frac{1}{3}}\right)$$

$$\theta \doteq 0.853$$

$$\text{or } \theta \doteq 48.9^\circ$$

Note: At some angle, θ , the length, L , will be at a minimum.



When $\theta \doteq 0.853$,

$$L \doteq \frac{3}{\sin(0.853)} + \frac{2}{\cos(0.853)}$$

$$L \doteq 7.02$$

$\therefore$ The length of the longest ladder will be approx. 7.02 m.

Ex 3. A fence 8 m tall runs parallel to a house at a distance of 3 m from the wall of a house. Determine the angle, θ , the shortest ladder will make from the foot of the ladder to the ground, if the ladder is to reach from the ground over the fence and to the wall of the house

Let L be the length of the shortest ladder,
where $L = l_1 + l_2$, and θ be the angle the ladder
forms with the ground.

$$L = l_1 + l_2 \quad \sin(\theta) = \frac{8}{l_1} \quad \cos(\theta) = \frac{3}{l_2}$$

$$L = \frac{8}{\sin(\theta)} + \frac{3}{\cos(\theta)} \quad \xleftarrow{\text{sub}} \quad l_1 = \frac{8}{\sin(\theta)} \quad l_2 = \frac{3}{\cos(\theta)}$$

$$= 8[\sin(\theta)]^{-1} + 3[\cos(\theta)]^{-1}$$

$$L' = -8[\sin(\theta)]^{-2}[\cos(\theta)] - 3[\cos(\theta)]^{-2}[-\sin(\theta)]$$

$$= \frac{-8\cos(\theta)}{\sin^2(\theta)} + \frac{3\sin(\theta)}{\cos^2(\theta)}$$

$$\text{critical \#s} \Rightarrow L' = 0$$

$$\frac{8\cos(\theta)}{\sin^3(\theta)} = \frac{3\sin(\theta)}{\cos^2(\theta)}$$

$$8\cos^3(\theta) = 3\sin^3(\theta)$$

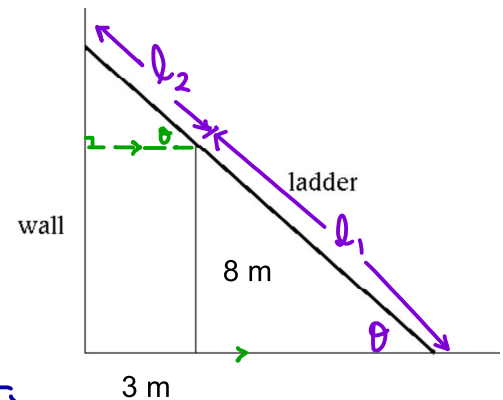
$$\tan^3(\theta) = \frac{8}{3}$$

$$\tan(\theta) = \left(\frac{8}{3}\right)^{\frac{1}{3}}, \quad 0 < \theta < \frac{\pi}{2}$$

$$\theta \doteq 0.946$$

$$\text{or } \theta \doteq 54.2^\circ$$

$\therefore$ The shortest ladder will make an angle of
approx. 0.946 radians with the ground.



Related Rates Problems with Trigonometric Functions

General Method for solving related rates problems:

1. Draw a picture and give names to all quantities involved.
2. Which rate do we want? Which rate do we know?
3. Find relationships between the quantities that are true for all times in the problem. The goal is to have a complete chain between the quantities whose rate we know and the quantity whose rate we want.
4. Take a derivative with respect to time of both sides of every relationship.
5. Plug in the specific values for the moment in time we care about. Do not do this step early.
6. Solve for the rate we want.

1. A lighthouse is located on a small island 3 km away from the nearest point P on a straight shoreline. If the lighthouse beacon rotates at a constant rate of 4 revolutions per minute, how fast is the beam of light moving along the shoreline when it is 1 km from the nearest point P ?

Let x be the distance along the shoreline, θ be the angle of the beam and t be the time.

Givens: $\frac{d\theta}{dt} = 4 \text{ rev/min}$
 $= 8\pi \text{ rad/min}$

Find: $\frac{dx}{dt} = ?$ when $x = 1 \text{ km}$

At moment when $x=1$,

$$\cos(\theta) = \frac{3}{\sqrt{3^2 + 1^2}}$$

$$= \frac{3}{\sqrt{10}}$$

$$\therefore \sec(\theta) = \frac{\sqrt{10}}{3}$$

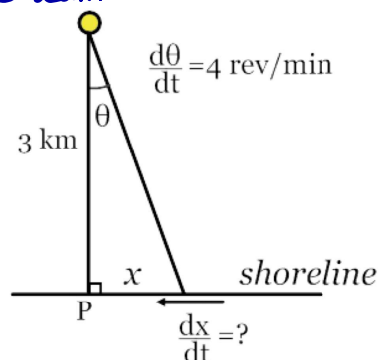
$$\tan(\theta) = \frac{x}{3}$$

$$\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{3} \frac{dx}{dt}$$

At the instant, $x=1 \Rightarrow \sec(\theta) = \frac{\sqrt{10}}{3}$:

$$\left(\frac{\sqrt{10}}{3}\right)^2 \cdot (8\pi) = \frac{1}{3} \cdot \frac{dx}{dt}$$

$$\therefore \frac{dx}{dt} = \frac{80\pi}{3} \text{ or } \approx 83.8$$



$\therefore$ The beam is moving at approx. 83.8 km/min along the shoreline.

2. If the height of an isosceles triangle with base 2m changes at a rate 3 m/s, how quickly is the angle opposite the base changing when $h = \sqrt{3} \text{ m}$?

Let h be the height of the triangle, θ be the angle opposite the base and t be the time.

Givens: $\frac{dh}{dt} = 3 \text{ m/s}$

Find: $\frac{d\theta}{dt} = ?$ when $h = \sqrt{3} \text{ m}$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1}{h}$$

$$\frac{1}{2} \sec^2\left(\frac{\theta}{2}\right) \cdot \frac{d\theta}{dt} = -\frac{1}{h^2} \cdot \frac{dh}{dt}$$

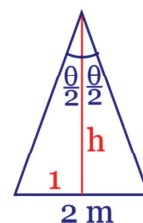
At the instant, $h = \sqrt{3} \Rightarrow \frac{dh}{dt} = 3, \sec\left(\frac{\theta}{2}\right) = \frac{2}{\sqrt{3}}$:

$$\frac{1}{2} \left(\frac{2}{\sqrt{3}}\right)^2 \cdot \frac{d\theta}{dt} = -\frac{1}{(\sqrt{3})^2} (3)$$

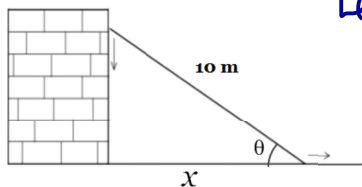
$$\frac{2}{3} \cdot \frac{d\theta}{dt} = -1$$

$$\frac{d\theta}{dt} = -\frac{3}{2}$$

$\therefore$ The angle opposite the base is decreasing at a rate of $\frac{3}{2} \text{ rad/s}$.



3. A 10-m ladder leans against a house on flat ground. The house is to the left of the ladder. The base of the ladder starts to slide away from the house at 2 m/s. At what rate is the angle between the ladder and the ground changing when the base is 8 m from the house?



Let x be the horizontal distance of the base of the ladder to the house, θ be the angle between the base of the ladder and the ground and t be the time.

Given: $\frac{dx}{dt} = 2 \text{ m/s}$

Find: $\frac{d\theta}{dt} = ?$ when $x = 8 \text{ m}$

When $x = 8$,

$$\sin(\theta) = \frac{\sqrt{10^2 - 8^2}}{10}$$

$$\sin(\theta) = \frac{3}{5}$$

$$\cos(\theta) = \frac{x}{10}$$

$$-\sin(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dx}{dt}$$

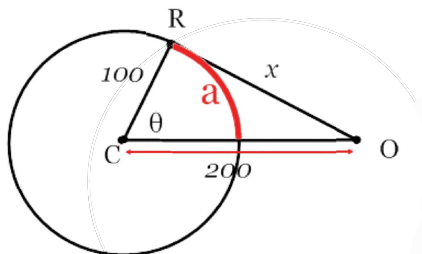
At the instant, $x = 8 \Rightarrow \frac{dx}{dt} = 2$, $\sin(\theta) = \frac{3}{5}$:

$$-\left(\frac{3}{5}\right) \cdot \frac{d\theta}{dt} = \frac{1}{10}(2)$$

$$\frac{d\theta}{dt} = -\frac{1}{3}$$

$\therefore$ The angle between the ladder and the ground is decreasing at a rate of $\frac{1}{3} \text{ rad/s}$.

4. A runner sprints around a circular track of radius 100 meters at a constant speed of 7 meters per second. An observer is standing 200 meters from the track's center. How fast is the distance between the runner and the observer changing when the distance between them is 200 meters?



Given: $\frac{da}{dt} = 7 \text{ m/s}$

Find: $\frac{dx}{dt} = ?$ when $x = 200 \text{ m}$

$$a = 100\theta \quad \frac{da}{dt} = 100 \frac{d\theta}{dt}$$

$$\frac{7}{100} = \frac{d\theta}{dt}$$

when $x = 200$,

$$(200)^2 = (200)^2 + (100)^2 - 2(100)(200)\cos(\theta)$$

$$\frac{200^2 - 200^2 - 100^2}{-2(100)(200)} = \cos(\theta)$$

$$\frac{1}{4} = \cos(\theta)$$

$$\therefore \sin(\theta) = \frac{\sqrt{4^2 - 1^2}}{4} = \frac{\sqrt{15}}{4}$$



Let x be the distance between the runner and the observer, θ be the angle at the center formed by the line of sight to the runner and the line of sight to the observer, a be the distance the runner is travelling (arclength) and t be the time.

$$x^2 = (200)^2 + (100)^2 - 2(200)(100)\cos(\theta)$$

$$2x \frac{dx}{dt} = 40000 \sin(\theta) \frac{d\theta}{dt}$$

At the instant, $x = 200 \Rightarrow \frac{d\theta}{dt} = \frac{7}{100}$, $\sin(\theta) = \frac{\sqrt{15}}{4}$:

$$2(200) \frac{dx}{dt} = 40000 \left(\frac{\sqrt{15}}{4}\right) \left(\frac{7}{100}\right)$$

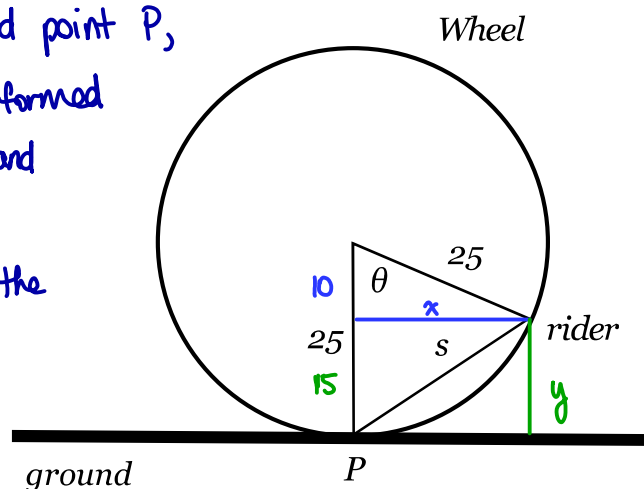
$$\frac{dx}{dt} = \frac{7\sqrt{15}}{4}$$

$$\approx 6.78$$

$\therefore$ The distance is increasing at a rate of approx. 6.78 m/s.

5. A Ferris wheel 50 meter in diameter makes 2 revolutions per minute. Assume that the wheel is tangent to the ground and let P be the point of tangency. At what rate is the distance between P and a rider changing, when she is 15 meter above the ground and going up?

Let s be the distance between the rider and point P,
 θ be the angle at the center of the wheel formed
 by the straight line distance to the rider and
 the straight line distance to P, x be the
 horizontal distance of the rider to P, y be the
 rider's height and t be the time.



Given: $\frac{d\theta}{dt} = 4\pi$ rad/min

Find: $\frac{ds}{dt} = ?$ when $y = 15$ m

when $y = 15$,

$$x = \sqrt{25^2 - 10^2}$$

$$= 5\sqrt{21}$$

$$s = \sqrt{(5\sqrt{21})^2 + (15)^2}$$

$$= 5\sqrt{30}$$

$$\sin(\theta) = \frac{x}{25}$$

$$\sin(\theta) = \frac{5\sqrt{21}}{25}$$

$$= \frac{\sqrt{21}}{5}$$

$$s^2 = (25)^2 + (25)^2 - 2(25)(25)\cos(\theta)$$

$$2s \frac{ds}{dt} = 1250 \sin(\theta) \frac{d\theta}{dt}$$

At the instant, $y = 15 \Rightarrow s = 5\sqrt{30}$, $\sin(\theta) = \frac{\sqrt{21}}{5}$, $\frac{d\theta}{dt} = 4\pi$:

$$2(5\sqrt{30}) \frac{ds}{dt} = 1250 \left(\frac{\sqrt{21}}{5} \right) (4\pi)$$

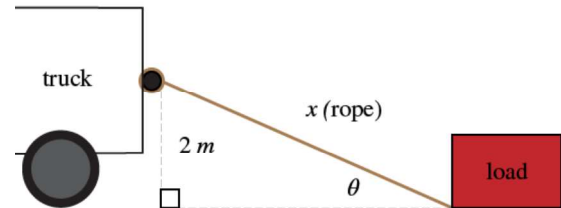
$$\frac{ds}{dt} = \frac{1000\sqrt{21}}{10\sqrt{30}} \pi$$

$$\frac{ds}{dt} \doteq 83.7\pi$$

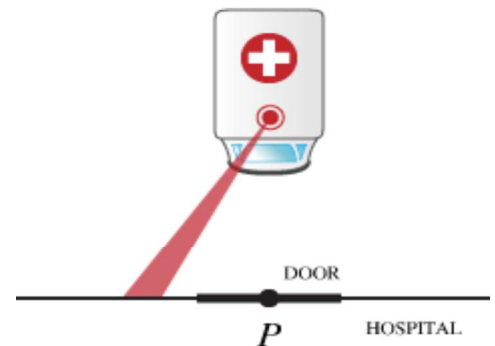
$\therefore$ The distance between the rider and point P is increasing at a rate of approx. 83.7π m/min.

Related Rates Practice

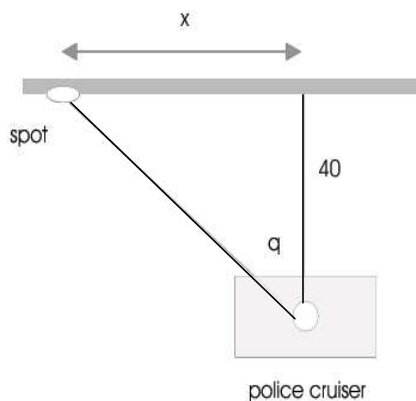
1. A rotating beacon is located 2 km out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 km along the shore from the point A? [Ans. 80π]
2. A winch on the back of a stationary tow truck is used to drag a large load along the ground. The winch is located 2 m above the ground and cranks the cable in at a rate of 0.75 m/s. At what rate is the angle between the cable and the ground changing when the angle is equal to $\frac{\pi}{4}$? [Ans. $\frac{3\sqrt{2}}{16}$ rad / sec]



3. Two sides of a triangle have lengths 8 cm and 15 cm. The angle, θ , between them is increasing at a rate of 160 radians per second. It follows that the length of the third side, c , is also increasing with respect to time.
 - (a) Use the cosine law to find an equation relating the angle θ to the length of the third side, c . [Ans. $c^2 = 289 - 240\cos(\theta)$]
 - (b) Find the length of the third side, c , when $\theta = \frac{\pi}{2}$. [Ans. $c = 17$]
 - (c) At what rate is the length of the third side, c , changing when $\theta = \frac{\pi}{2}$? [Ans. $\frac{2}{17}$ cm / s]
4. The light on the top of an ambulance rotates and makes one revolution every 5 seconds. The light runs along a wall on the side of the hospital as it rotates. If the light is situated 10 metres from the wall, then how fast is the spot of light moving along the hospital wall when the spot is 3 metres from the centre of the door, located at point P? [Ans. $\frac{109\pi}{25}$ m / s]



5. A spotlight on the top of a police cruiser makes one revolution per second. The spotlight is 40 meter from a long straight wall. At what rate is the spot of light moving across the wall at the instant when the beam makes an angle of $\frac{\pi}{6}$ with the wall? [Ans. 320π m/s]



Related Rates Practice-Solutions

1. A rotating beacon is located 2 km out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 km along the shore from the point A? [Ans. 80π]

$$\frac{d\theta}{dt} = 10 \text{ rev / min} = 20\pi \text{ rad / min}$$

$$\frac{dx}{dt} = ? \text{ when } x = 2 \text{ km}$$

$$\tan(\theta) = \frac{x}{2}$$

$$\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{2} \cdot \frac{dx}{dt}$$

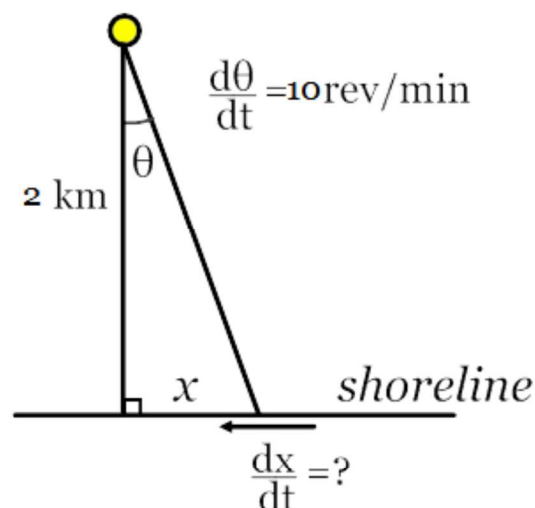
$$\text{when } x = 2, \cos(\theta) = \frac{2}{\sqrt{2^2 + 2^2}}$$

$$\cos(\theta) = \frac{2}{\sqrt{2}}$$

$$\sec(\theta) = \frac{\sqrt{2}}{2}$$

$$\text{Plug in: } \left(\frac{\sqrt{2}}{2}\right)^2 \cdot 20\pi = \frac{1}{2} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = 80\pi \text{ km / min}$$



2. A winch on the back of a stationary tow truck is used to drag a large load along the ground. The winch is located 2 m above the ground and cranks the cable in at a rate of 0.75 m/s. At what rate is the angle between the cable and the ground changing when the

angle is equal to $\frac{\pi}{4}$? [Ans. $\frac{3\sqrt{2}}{16}$ rad / sec]

$$\frac{dx}{dt} = -0.75 \text{ m / s}$$

$$\frac{d\theta}{dt} = ? \text{ when } \theta = \frac{\pi}{4}$$

$$\sin(\theta) = \frac{2}{x}$$

$$x = 2\csc(\theta)$$

$$\frac{dx}{dt} = -2\csc(\theta)\tan(\theta) \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{2} \frac{\sin(\theta)}{\tan(\theta)} \left(\frac{dx}{dt}\right)$$

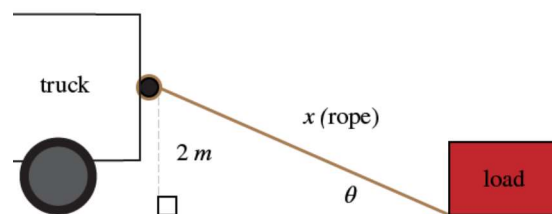
$$= -\frac{1}{2} \cos(\theta) \left(\frac{dx}{dt}\right)$$

Substituting the values $\theta = \frac{\pi}{4}$ and $\frac{dx}{dt} = -0.75 = -\frac{3}{4}$, we obtain

$$\frac{d\theta}{dt} = -\frac{1}{2} \cos\left(\frac{\pi}{4}\right) \left(-\frac{3}{4}\right)$$

$$= \frac{3}{8} \left(\frac{\sqrt{2}}{2}\right)$$

$$= \frac{3\sqrt{2}}{16} \text{ rad / sec}$$



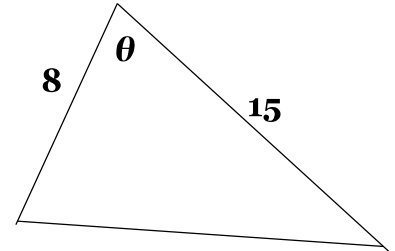
Related Rates Practice-Solutions

3. Two sides of a triangle have lengths 8 cm and 15 cm. The angle, θ , between them is increasing at a rate of $\frac{1}{60}$ radians per second. It follows that the length of the third side, c , is also increasing with respect to time.

(a) Use the cosine law to find an equation relating the angle θ to the length of the third side, c . [Ans. $c^2 = 289 - 240\cos(\theta)$]

$$c^2 = 8^2 + 15^2 - 2(8)(15)\cos(\theta)$$

$$c^2 = 289 - 240\cos(\theta)$$



(b) Find the length of the third side, c , when $\theta = \frac{\pi}{2}$. [Ans. $c=17$]

$$\theta = \frac{\pi}{2} \rightarrow \cos(\theta) = 0$$

$$c^2 = 289 - 240\cos(\theta)$$

$$c^2 = 289$$

$$c = 17$$

(c) At what rate is the length of the third side, c , changing when $\theta = \frac{\pi}{2}$? [Ans. $\frac{2}{17}$ cm / s]

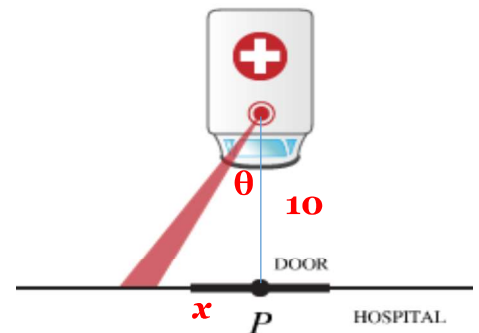
$$\text{when } \theta = \frac{\pi}{2}, \frac{d\theta}{dt} = \frac{1}{60} \text{ rad / sec}$$

$$2c \frac{dc}{dt} = 240\sin(\theta) \left(\frac{d\theta}{dt} \right)$$

$$17 \frac{dc}{dt} = 120(1) \left(\frac{1}{60} \right)$$

$$\frac{dc}{dt} = \frac{2}{17} \text{ cm / sec}$$

4. The light on the top of an ambulance rotates and makes one revolution every 5 seconds. The light runs along a wall on the side of the hospital as it rotates. If the light is situated 10 metres from the wall, then how fast is the spot of light moving along the hospital wall when the spot is 3 metres from the centre of the door, located at point P ? [Ans. $\frac{109\pi}{25}$ m / s]



$$\frac{d\theta}{dt} = \frac{1}{5} \text{ rev / sec} = \frac{2\pi}{5} \text{ rad / sec}$$

$$\frac{dx}{dt} = ? \text{ when } x = 3 \text{ m}$$

$$\tan(\theta) = \frac{x}{10}$$

$$\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dx}{dt}$$

$$\text{when } x = 3, \cos(\theta) = \frac{10}{\sqrt{10^2 + 3^2}}$$

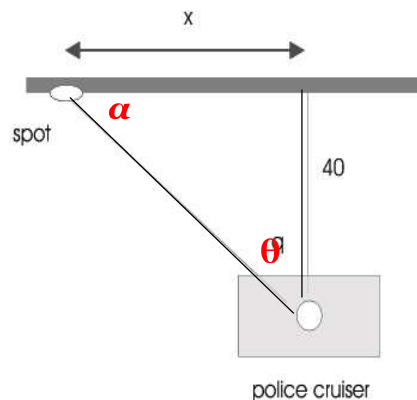
$$\cos(\theta) = \frac{10}{\sqrt{109}} \Rightarrow \sec(\theta) = \frac{\sqrt{109}}{10}$$

$$\text{Plug in: } \left(\frac{\sqrt{109}}{10} \right)^2 \cdot \frac{2\pi}{5} = \frac{1}{10} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = \frac{109\pi}{25} \text{ m / sec}$$

Related Rates Practice-Solutions

5. A spotlight on the top of a police cruiser makes one revolution per second. The spotlight is 40 meter from a long straight wall. At what rate is the spot of light moving across the wall at the instant when the beam makes an angle of $\frac{\pi}{6}$ with the wall? [Ans. 320π m/s]



$$\frac{d\theta}{dt} = 1 \text{ rev / sec} = 2\pi \text{ rad / sec}$$

$$\frac{dx}{dt} = ? \text{ when } \alpha = \frac{\pi}{6} \rightarrow \theta = \frac{\pi}{3}$$

$$\tan(\theta) = \frac{x}{40}$$

$$\sec^2(\theta) \frac{d\theta}{dt} = \frac{1}{40} \cdot \frac{dx}{dt}$$

$$\text{when } \theta = \frac{\pi}{3}, \cos(\theta) = \frac{1}{2}$$

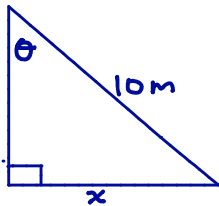
$$\sec(\theta) = 2$$

$$\text{Plug in: } (2)^2 \cdot 2\pi = \frac{1}{40} \cdot \frac{dx}{dt}$$

$$\frac{dx}{dt} = 320\pi \text{ m / sec}$$

Related Rates & Optimization Involving Trigonometric Derivatives

Example 1: A ladder 10m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2m/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$ rad?



$$\frac{dx}{dt} = 2 \text{ m/s}$$

$$\frac{d\theta}{dt} = ? \text{ when } \theta = \frac{\pi}{4}$$

Let x be the distance from the bottom of the ladder to the wall, θ be the angle between the top of the ladder and the wall and t be the time in seconds.

$$\sin \theta = \frac{x}{10}$$

Differentiate w.r.t. time:

$$\cos \theta \frac{d\theta}{dt} = \frac{1}{10} \frac{dx}{dt}$$

$$\cos \frac{\pi}{4} \frac{d\theta}{dt} = \frac{1}{10} (2)$$

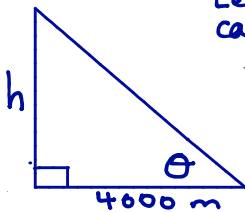
$$\frac{d\theta}{dt} = \frac{1}{5} \div \cos \frac{\pi}{4}$$

$$\frac{d\theta}{dt} = \frac{1}{5} (\sqrt{2})$$

$$\frac{d\theta}{dt} = \frac{\sqrt{2}}{5} \approx 0.283$$

$\therefore$ The angle is changing at a rate of $\approx 0.283 \text{ rad/s}$.

Example 2: A television camera is positioned 4000 m from the base of a rocket launching pad. If the camera is aimed at the rocket that is travelling at a speed of 600m/s, determine how fast the camera's angle is changing the moment the rocket has reached a height of 3000m.



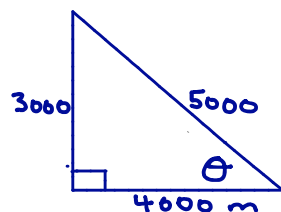
$$\frac{dh}{dt} = 600 \text{ m/s}$$

$$\frac{d\theta}{dt} = ? \text{ when } h = 3000 \text{ m}$$

$$\tan \theta = \frac{3000}{4000}$$

$$\tan \theta = \frac{3}{4}$$

$$\cos \theta = \frac{4}{5}$$



$$\tan \theta = \frac{h}{4000}$$

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{4000} \frac{dh}{dt}$$

$$\frac{d\theta}{dt} = \frac{\frac{1}{4000} \cdot \frac{dh}{dt}}{\frac{1}{\cos^2 \theta}}$$

$$= \frac{1}{4000} (600) \left(\frac{4}{5}\right)^2$$

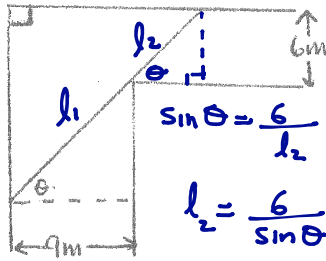
$$= \frac{3}{20} \cdot \left(\frac{16}{25}\right)$$

$$= \frac{12}{125}$$

$$\approx 0.0962$$

$\therefore$ The camera's angle is changing at a rate of $\approx 0.0962 \text{ rad/s}$.

Example 3: A steel pipe is being carried down a hallway 9 m wide. At the end of the hall there is a right angled turn into a narrower hallway 6 m wide. What is the length of the longest pipe that can be carried horizontally around the corner?



$$\cos \theta = \frac{9}{l_1}$$

$$l_1 = \frac{9}{\cos \theta}$$

$$0 < \theta < \frac{\pi}{2}$$

Note: At some angle, θ , the length, L , will be a minimum - the required length of the longest ladder.

Let the length of the ladder be $L = l_1 + l_2$ and θ , the angle shown.

$$L = l_1 + l_2$$

$$L = \frac{9}{\cos \theta} + \frac{6}{\sin \theta} \rightarrow L = 9(\cos \theta)^{-1}$$

$$L'(\theta) = -9(\cos \theta)^{-2}(-\sin \theta) - 6(\sin \theta)^{-2} \cos \theta$$

$$= \frac{9 \sin \theta}{\cos^2 \theta} - \frac{6 \cos \theta}{\sin^2 \theta}$$

$$\text{critical \#s} \Rightarrow L'(\theta) = 0$$

$$\frac{9 \sin^3 \theta - 6 \cos^3 \theta}{\cos^2 \theta \sin^2 \theta} = 0$$

$$9 \sin^3 \theta = 6 \cos^3 \theta$$

$$\tan^3 \theta = \frac{6}{9}$$

$$\tan^3 \theta = \frac{2}{3}$$

$$\tan \theta = \left(\frac{2}{3}\right)^{\frac{1}{3}}$$

$$\theta = \tan^{-1} \left(\frac{2}{3}\right)^{\frac{1}{3}}$$

$$\theta \approx 41.1^\circ \text{ or } 0.718 \text{ rad}$$

$$\therefore L = \frac{9}{\cos 0.718} + \frac{6}{\sin 0.718}$$

$$L \approx 21.1$$

2nd derivative test:

$$L''(\theta) = \frac{9 \cos \theta \cos^2 \theta - 2 \cos \theta (-\sin \theta)(9 \sin \theta)}{\cos^4 \theta} - \frac{[6(-\sin \theta) \sin^2 \theta - 2 \sin \theta \cos \theta (6 \cos \theta)]}{\sin^4 \theta}$$

$$= \frac{9 \cos^3 \theta + 18 \sin^2 \theta \cos \theta}{\cos^4 \theta} + \frac{6 \sin^3 \theta + 12 \sin \theta \cos^2 \theta}{\sin^4 \theta}$$

$$= \frac{9 \cos \theta (\cos^2 \theta + 2 \sin^2 \theta)}{\cos^4 \theta} + \frac{6 \sin \theta (\sin^2 \theta + 2 \cos^2 \theta)}{\sin^4 \theta}$$

$$= \frac{9(\cos^2 \theta + 2 \sin^2 \theta)}{\cos^3 \theta} + \frac{6(\sin^2 \theta + 2 \cos^2 \theta)}{\sin^3 \theta}$$

$$L''(41.1^\circ) > 0 \quad \text{min value}$$

$$\sin 41.1^\circ > 0, \cos 41.1^\circ > 0$$

$\therefore$ The length of the longest pipe is ≈ 21.1 m.

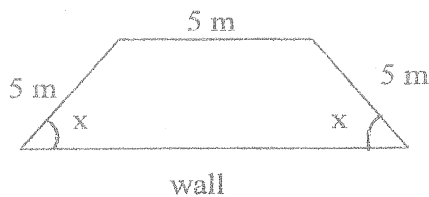
Application of derivatives of trigonometric functions

1. The height, h metres, of a Ferris wheel above the ground at time t seconds is given by
 $h = 20 + 20 \sin\left(\frac{t + \pi}{4}\right)$. Find the rate at which its height is changing when $t = 10$ seconds.

2. A farmer has three pieces of fencing, each of length 5 metres, and wishes to enclose a pen, as in the diagram. A long wall, which is already in place, will comprise the fourth side of the pen.

a) Show that the area of the pen is $[25 \sin x(1 + \cos x)] \text{ m}^2$.

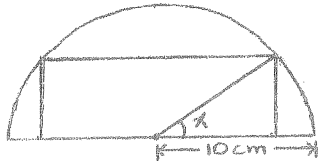
b) Calculate the maximum value of this area.



3. A rectangle is drawn inside a semicircle of base radius 10 cm as in the diagram.

a) Show that the area of the rectangle is $200 \sin x \cos x \text{ cm}^2$.

b) Calculate the maximum value of this area.



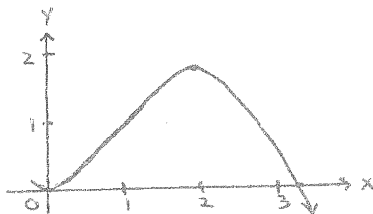
4. The diagram shows the graph of the function $g : x \rightarrow x \sin x$ for $-0.5 \leq x \leq 3.5$. The graph intersects the x -axis at the point A with coordinates $(a, 0)$, and B is the maximum point on this part of the graph.

a) Find the value of a .

b)i) Find $g'(x)$.

ii) Show that, at the maximum point B, x satisfies the equation $x + \tan x = 0$.

c) Show that the second derivative $g''(x) = -x \sin x + 2 \cos x$.



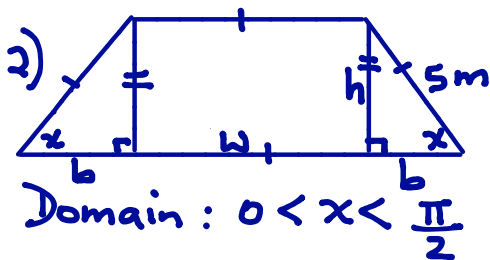
Application of derivatives of trigonometric functions

$$1) h = 20 + 20 \sin\left(\frac{t+\pi}{4}\right)$$

$$\frac{dh}{dt} = 20 \cos\left(\frac{t+\pi}{4}\right) \cdot \left(\frac{1}{4}\right)$$
$$= 5 \cos\left(\frac{t+\pi}{4}\right)$$

$$\left. \frac{dh}{dt} \right|_{t=10} = 5 \cos\left(\frac{10+\pi}{4}\right)$$
$$\approx -4.95$$

$\therefore$ The rate at which the height is changing is approx. -4.95 m/s .



Note: $\sin x = \frac{h}{5}$ $\cos x = \frac{b}{5}$
 $h = 5 \sin x$ $b = 5 \cos x$

a) Area of the pen = Area of the 2 $\triangle$ s + Area of rectangle

$$= 2 \left[\frac{1}{2} bh \right] + wh$$

or Area of a Trapezoid Formula:

$$A = \frac{1}{2}(a+b)h$$

$$= bh + wh$$

$$= (5 \cos x)(5 \sin x) + 5(5 \sin x)$$

$$= 25 \sin x \cos x + 25 \sin x$$

$$= 25 \sin x (\cos x + 1)$$

$\therefore$ The area of the pen is $[25 \sin x (\cos x + 1)] \text{ m}^2$

$$b) A(x) = 25 \sin x (1 + \cos x) \quad \text{or } A(x) = 25 \sin x + 25 \sin x \cos x$$

$$\begin{aligned} A'(x) &= 25 \cos x (1 + \cos x) + 25 \sin x (-\sin x) \\ &= 25 \cos x + 25 \cos^2 x - 25 \sin^2 x \\ &= 25 [\cos x + \cos^2 x - \sin^2 x] \\ &= 25 [\cos x + \cos^2 x - (1 - \cos^2 x)] \\ &= 25 [2 \cos^2 x + \cos x - 1] \\ &= 25 [(2 \cos x - 1)(\cos x + 1)] \end{aligned}$$

$$\text{critical \#s} \Rightarrow A'(x) = 0$$

$$2 \cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$

$$\cos x = \frac{1}{2}$$

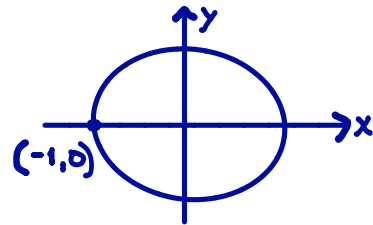
$$x = \frac{\pi}{3}$$

$$\therefore 0 < x < \frac{\pi}{2}$$

$$\cos x = -1$$

$$x = \pi$$

$$\text{inadmissible } 0 < x < \frac{\pi}{2}$$



$$\therefore A\left(\frac{\pi}{3}\right) = 25 \sin \frac{\pi}{3} \left(1 + \cos \frac{\pi}{3}\right)$$

$$= 25 \left(\frac{\sqrt{3}}{2}\right) \left(1 + \frac{1}{2}\right)$$

$$= 25 \left(\frac{\sqrt{3}}{2}\right) \left(\frac{3}{2}\right)$$

$$= \frac{75\sqrt{3}}{4} \quad \therefore \text{Area is } \frac{75\sqrt{3}}{4} \text{ m}^2$$

2nd derivative test :

$$A'(x) = 25 [2 \cos^2 x + \cos x - 1]$$

$$\begin{aligned} A''(x) &= 100 \cos x (-\sin x) - \sin x \\ &= -100 \cos x \sin x - \sin x \end{aligned}$$

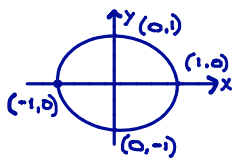
$$A''\left(\frac{\pi}{3}\right) < 0 \quad \therefore \text{max value at } x = \frac{\pi}{3}$$



a) Area of rectangle = $2(10 \cos x)(10 \sin x)$
 $A = 200 \sin x \cos x$

b) $A(x) = 100 \sin 2x$
 $A'(x) = 100(\cos 2x)(2)$
 $= 200 \cos 2x$

critical #s $\Rightarrow A'(x) = 0$
 $200 \cos 2x = 0$



$\cos 2x = 0$

$2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

or $A(x) = 200 \sin x \cos x$

$A'(x) = 200(\cos x \cos x - \sin x \sin x)$
 $= 200(\cos^2 x - \sin^2 x)$
 $= 200(1 - 2\sin^2 x)$ or $200(2\cos^2 x - 1)$

critical #s $\rightarrow A'(x) = 0$

$0 = 200(1 - 2\sin^2 x)$

$\frac{1}{2} = \sin^2 x$

$\pm \frac{1}{\sqrt{2}} = \sin x$

consider Q1 only $\because 0 < x < \frac{\pi}{2}$

$\therefore x = \frac{\pi}{4}$

$\therefore A\left(\frac{\pi}{4}\right) = 200 \sin \frac{\pi}{4} \cos \frac{\pi}{4}$
 $= 200\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$
 $= 100$

$\therefore$ Area is 100 cm^2

2nd derivative test:

$A'(x) = 200 \cos 2x$

$A''(x) = -400 \sin 2x$

$A''\left(\frac{\pi}{4}\right) < 0 \quad \cap \quad \text{max. value}$

4) a) Method ①:

• Graph $y = x \sin x$, $-0.5 \leq x \leq 3.5$ using GDC
 $-0.5 \leq y \leq 2$

• Find the zeros

$$x = 0 \text{ or } x \approx 3.1415927$$

$x = \pi$ $\therefore a = \pi$ $\therefore$ Point A is not on the origin.

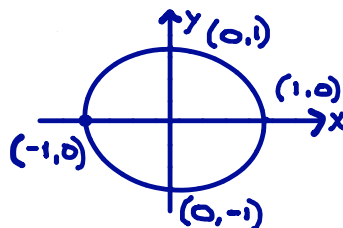
Method ②

$$x \sin x = 0$$

$$x = 0 \text{ or } \sin x = 0$$

$$x = 0 \text{ or } x = \pi$$

$$\therefore a = \pi$$



b) i) $g(x) = x \sin x$

$$g'(x) = \sin x + x \cos x$$

ii) critical #s $\Rightarrow g'(x) = 0$

$$\sin x + x \cos x = 0$$

$$x \cos x = -\sin x$$

$$x = \frac{-\sin x}{\cos x}$$

$$x = -\tan x$$

$$x + \tan x = 0$$

$\therefore x + \tan x = 0$ is satisfied

c) $g'(x) = \sin x + x \cos x$

$$g''(x) = \cos x + \cos x - x \sin x$$
$$= 2 \cos x - x \sin x$$

$\rightarrow$ OR using GDC:

① $y_1 = x \sin x \rightarrow$ graph

② Find the maximum

$\hookrightarrow$ 2nd trace $\rightarrow$ calculate

$$x \approx 2.0287578$$

OR using the GDC:

① $y_1 = x$ and $y_2 = -\tan x$

② And pt. of intersection:

$$x \approx 2.0287578$$

$$\therefore LS = x + \tan x$$

$$LS \approx (2.0287578) +$$

$$\tan(2.0287578)$$

$$\approx 0$$

$$\therefore LS = RS$$

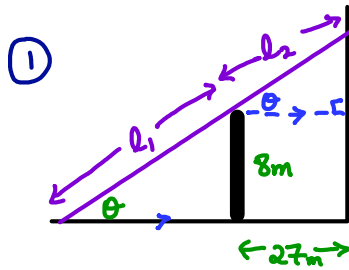
Applications of Trigonometric Functions

1. A wall of height 8 m stands parallel to and 27 in from a tall building. A ladder with its foot on the ground is to pass over the top of the wall and lean on the side of the building. What angle will the shortest such ladder make with the ground?
2. A kite 40 m above the ground moves horizontally at the rate of 3 m/s. At what rate is the angle between the string and the horizontal decreasing when 80 m of string has been released?
3. A revolving beacon is situated 925 m from a straight shore. It turns at 2 rev/min.
 - a) How fast does the beam sweep along the shore at its nearest point?
 - b) How fast does it sweep along the shore at a point 1275 m from the nearest point?
4. Two sides of a triangle are six and eight meters in length. The angle between them decreases at the rate of 0.035 radian/s. Find the rate at which the area is decreasing when the angle between the sides of fixed length is $\frac{\pi}{6}$.
5. A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 m/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$?
6. The base of an isosceles triangle is 20 cm and the altitude is increasing at the rate of 1 cm/min. At what rate is the base angle increasing when the area is 100 cm²?
7. A vehicle moves along a straight path with a speed of 4 m/s. A searchlight is located on the ground 20 m from the path and is kept focused on the vehicle. At what rate (in rad/s) is the searchlight rotating when the vehicle is 15 m from the point on the path closest to the searchlight?

Answers:

1. 0.588
2. 0.05 rad/s
3. a) 11624 m/min; b) 33708 m/min
4. 0.727 m²/s
5. $\frac{\sqrt{2}}{5}$ rad/s
6. 0.05 rad/min
7. 0.128 rad/s

Applications of Trigonometric Functions solutions



Let L be the length of the ladder and
 $L = l_1 + l_2$, let θ be the angle the ladder makes with the ground

$$\sin \theta = \frac{8}{l_1}$$

$$\cos \theta = \frac{27}{l_2}$$

$$l_1 = \frac{8}{\sin \theta}$$

$$l_2 = \frac{27}{\cos \theta}$$

$$L = \frac{8}{\sin \theta} + \frac{27}{\cos \theta}$$

$$L = 8(\sin \theta)^{-1} + 27(\cos \theta)^{-1}$$

$$L' = \frac{-8 \cos \theta}{\sin^2 \theta} + \frac{-27(-\sin \theta)}{\cos^2 \theta}$$

$$0 = \frac{-8 \cos \theta}{\sin^2 \theta} + \frac{27 \sin \theta}{\cos^2 \theta}$$

$$\frac{8 \cos \theta}{\sin^2 \theta} = \frac{27 \sin \theta}{\cos^2 \theta}$$

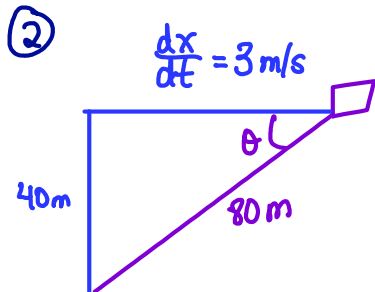
$$8 \cos^3 \theta = 27 \sin^3 \theta$$

$$\frac{8}{27} = \tan^3 \theta$$

$$\frac{2}{3} = \tan \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\theta \doteq 0.58800$$

$\therefore$ The shortest ladder will make
 an angle of approx. 0.588 rad



Let θ be the angle the string forms with the horizontal,
 x be the horizontal distance of the kite and t be
 the time in seconds.

$$\tan \theta = \frac{40}{x}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = -40x^{-2} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{-40}{x^2} \cdot \cos^2 \theta \cdot \frac{dx}{dt}$$

$$x = \sqrt{80^2 - 40^2}$$

$$= 40\sqrt{3}$$

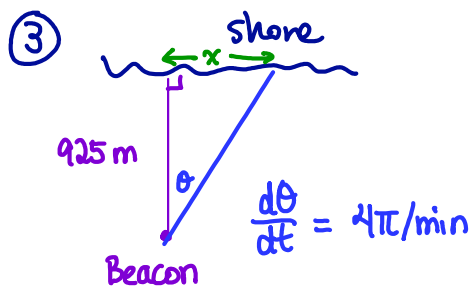
$$\frac{d\theta}{dt} = \frac{-40}{(40\sqrt{3})^2} \cdot \frac{\sqrt{3}}{2} \cdot 3$$

$$= \frac{-120\sqrt{3}}{9600}$$

$$= -\frac{\sqrt{3}}{80} \text{ or approx. } -0.0217$$

$\therefore$ The angle is decreasing
 at approx. 0.0217 rad/s.

Applications of Trigonometric Functions solutions



Let θ be the angle the beam of light from the beacon forms with a perpendicular line to the shore, x be the horizontal distance of the light from the perpendicular line and t be time in minutes

a) At closest point,
 $\theta = 0$
 $\therefore \cos \theta = 1$

$$\tan \theta = \frac{x}{925}$$
$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{925} \cdot \frac{dx}{dt}$$
$$1 (4\pi) = \frac{1}{925} \cdot \frac{dx}{dt}$$
$$11623.9 \doteq \frac{dx}{dt}$$

$\therefore$ The beacon is sweeping at approx. 11600 m/min at its nearest point.

b) $x = 1275$ m

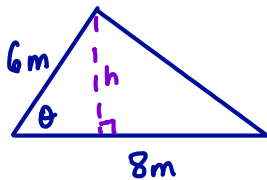
$$\begin{aligned} \text{hypotenuse} &= \sqrt{925^2 + 1275^2} \\ &= \sqrt{2481250} \\ &= 25\sqrt{3970} \\ \cos \theta &= \frac{925}{25\sqrt{3970}} \\ &= \frac{37}{\sqrt{3970}} \end{aligned}$$

$$\tan \theta = \frac{x}{925}$$
$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{925} \cdot \frac{dx}{dt}$$
$$\left(\frac{\sqrt{3970}}{37}\right)^2 \cdot 4\pi = \frac{1}{925} \cdot \frac{dx}{dt}$$
$$33708.44 \doteq \frac{dx}{dt}$$

$\therefore$ The beacon is sweeping at approx. 33700 m/min at a point 1275 m from the nearest point.

Applications of Trigonometric Functions solutions

④



Let θ be the angle between the 2 fixed sides, h be the height of the triangle, A be the area of the triangle and t be the time in seconds

$$\frac{d\theta}{dt} = -0.035 \text{ rad/s}$$

$$h = 6 \sin \theta$$

$$\text{If } \theta = \frac{\pi}{6},$$
$$\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$A = \frac{8(6 \sin \theta)}{2}$$

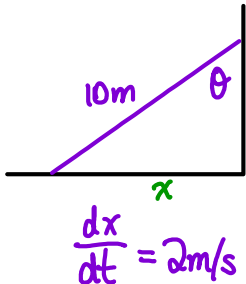
$$A = 24 \sin \theta$$

$$\frac{dA}{dt} = 24 \cos \theta \cdot \frac{d\theta}{dt}$$

$$\frac{dA}{dt} = 24 \left(\frac{\sqrt{3}}{2} \right) \cdot (-0.035)$$
$$\doteq -0.727$$

$\therefore$ The area is decreasing at a rate of approx. $0.727 \text{ m}^2/\text{s}$.

⑤



Let the angle between the ladder and the wall be θ , x be the horizontal distance of the base of the ladder to the wall and t be time in seconds.

$$\sin \theta = \frac{x}{10}$$

$$\cos \theta \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dx}{dt}$$

$$\frac{1}{\sqrt{2}} \cdot \frac{d\theta}{dt} = \frac{1}{10} (2)$$

$$\frac{d\theta}{dt} = \frac{\sqrt{2}}{5}$$

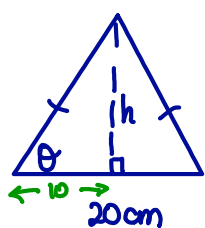
$\therefore$ The angle is increasing at a rate of $\frac{\sqrt{2}}{5} \text{ rad/s}$.

$$\text{At } \theta = \frac{\pi}{4},$$

$$\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

Applications of Trigonometric Functions solutions

⑥



Let θ be the base angle, h be the altitude, A be the area and t be the time in minutes.

$$\tan \theta = \frac{h}{10}$$

$$\frac{dh}{dt} = 1 \text{ cm/min}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot \frac{dh}{dt}$$

$$A = \frac{20h}{2}$$

$$(\sqrt{2})^2 \cdot \frac{d\theta}{dt} = \frac{1}{10} \cdot 1$$

$$A = 10h$$

$$\frac{d\theta}{dt} = \frac{1}{20}$$

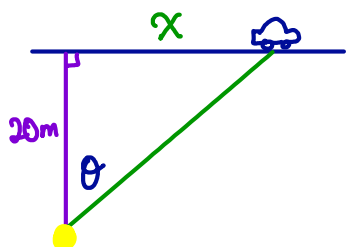
when $A = 100 \text{ cm}^2$,

$$h = 10 \text{ cm}$$

$$\text{and } \theta = \frac{\pi}{4}$$

$\therefore$ The base angle is increasing at a rate of $\frac{1}{20} \text{ rad/min}$

⑦



$$\frac{dx}{dt} = 4 \text{ m/s}$$

Let θ be the angle of the searchlight from the perpendicular line of the beam to the path, x be the distance the car travels from the closest point and t be the time in seconds

$$\tan \theta = \frac{x}{20}$$

At $x = 15 \text{ m}$,

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{20} \cdot \frac{dx}{dt}$$

$$\text{hypotenuse} = \sqrt{20^2 + 15^2}$$

$$= 25$$

$$\left(\frac{5}{4}\right)^2 \cdot \frac{d\theta}{dt} = \frac{1}{20} \cdot 4$$

$$\cos \theta = \frac{20}{25}$$

$$= \frac{4}{5}$$

$$\frac{d\theta}{dt} = \frac{16}{125}$$

$\therefore$ The searchlight is rotating at a rate of $\frac{16}{125}$ (or 0.128) rad/s when the car is 15 m from the point on the path closest to the searchlight.

Practise, Apply, Solve: Derivatives of trigonometric functions

- Find the equation of the tangent to the curve $y = x + \sin x$ at the point where $x = \frac{\pi}{3}$.
- Find the coordinates of the two points on the curve $y = \sin x(2 \cos x + 1)$, in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, where the gradient is $-\frac{1}{2}$.
- Given that $y = \frac{1}{\cos x} + \frac{1}{\sin x}$, show that $\frac{dy}{dx} = \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x}$.
Hence find the coordinates of the point on the curve $y = \frac{1}{\cos x} + \frac{1}{\sin x}$, in the range $0 \leq x \leq \pi$, where the gradient is zero.
- Find and classify the stationary values on each of the following curves in the range $0 \leq x \leq 2\pi$. Sketch each curve using your GDC.
 - $y = x + 2 \cos x$
 - $y = \frac{\sin x}{2 - \sin x}$
- The function $f(x) = a \sin x + b$ has a derivative $f'(x) = 3 \cos x$. The point $(0, 2)$ lies on the graph of the function. Find the values of a and b .
- The point $P\left(\frac{1}{3}, 0\right)$ lies on the graph of the equation $y = \sin(3x - 1)$.
 - Find $\frac{dy}{dx}$.
 - Hence find the gradient of the tangent to the graph at P .
 - How many tangents parallel to this one are there between $x = 0$ and $x = 2\pi$?
- A right-angled triangle has hypotenuse 15 cm and one angle x radians.
 - Find an expression for the perimeter of the triangle in terms of x .
 - Find an expression for the rate of change of the perimeter as x varies.
 - Hence find the value of x when this rate is greatest.

Answers:

- $9x - 6y + 3\sqrt{3} - \pi = 0$
- $\left(-\frac{\pi}{3}, -\sqrt{3}\right), \left(\frac{\pi}{3}, \sqrt{3}\right)$
- $\left(\frac{\pi}{4}, 2\sqrt{2}\right)$
- a) $\left(\frac{\pi}{6}, \frac{\pi}{6} + \sqrt{3}\right)$, max; $\left(\frac{5\pi}{6}, \frac{5\pi}{6} - \sqrt{3}\right)$, min b) $\left(\frac{\pi}{2}, 1\right)$, max; $\left(\frac{3\pi}{2}, -1\right)$, min
- a = 3, b = 2 6. a) $3 \cos(3x - 1)$ b) 3 c) 3 in total; at $x = \frac{1}{3}$ (given), $x = \frac{2\pi + 1}{3}$ and $x = \frac{4\pi + 1}{3}$
- a) $15(1 + \cos x + \sin x)$ b) $15(\cos x - \sin x)$ c) $\frac{\pi}{4}$

PAS solutions

①

$$\textcircled{2} y = \sin x (2 \cos x + 1), \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$y' = \cos x (2 \cos x + 1) + \sin x (-2 \sin x)$$

$$-\frac{1}{2} = 2 \cos^2 x + \cos x - 2 \sin^2 x$$

$$0 = 2 \cos^2 x + \cos x - 2(1 - \cos^2 x) + \frac{1}{2}$$

$$0 = 2 \cos^2 x + \cos x - 2 + 2 \cos^2 x + \frac{1}{2}$$

$$0 = 4 \cos^2 x + \cos x - \frac{3}{2}$$

$$0 = 8 \cos^2 x + 2 \cos x - 3$$

$$0 = (4 \cos x + 3)(2 \cos x - 1)$$

$$\therefore \cos x = -\frac{3}{4} \text{ or } \cos x = \frac{1}{2}$$

out of domain

$$\alpha = \frac{\pi}{3}$$

$$x_1 = \frac{\pi}{3}$$

$$x_2 = -\frac{\pi}{3}$$

$$\begin{aligned} \text{At } x_1, y &= \sin\left(\frac{\pi}{3}\right) \left[2 \cos\left(\frac{\pi}{3}\right) + 1\right] \\ &= \frac{\sqrt{3}}{2} \left[2\left(\frac{1}{2}\right) + 1\right] \\ &= \sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{At } x_2, y &= \sin\left(-\frac{\pi}{3}\right) \left[2 \cos\left(-\frac{\pi}{3}\right) + 1\right] \\ &= -\frac{\sqrt{3}}{2} \left[2\left(\frac{1}{2}\right) + 1\right] \\ &= -\sqrt{3} \end{aligned}$$

$$\therefore \text{Points } \left(\frac{\pi}{3}, \sqrt{3}\right) \text{ and } \left(-\frac{\pi}{3}, -\sqrt{3}\right)$$

Warm up

1. Find $\frac{dy}{dx}$ for each of the following. **Completely** simplify your answers.

a) $y = \cos(x^2 + 1)^3 + \sin^2(2\pi)$

$$\begin{aligned} a) \quad y' &= -\sin(x^2+1)^3 \cdot 3(x^2+1)(2x) + 0 \\ &= -6x(x^2+1) \cdot \sin^2(x^2+1)^3 \end{aligned}$$

b) $y = \cos^2(\sqrt{x}) - \sin^2(\sqrt{x})$

$$\begin{aligned} y &= \cos(2\sqrt{x}) \\ y' &= -\sin(2\sqrt{x}) \cdot 2\left(\frac{1}{2\sqrt{x}}\right) \\ &= \frac{-\sqrt{x} \sin(2\sqrt{x})}{x} \end{aligned}$$

Alt solution b) $y' = 2\cos(\sqrt{x}) \cdot [-\sin(\sqrt{x})] \cdot \frac{1}{2\sqrt{x}} - 2\sin(\sqrt{x}) \cdot \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$

$$\begin{aligned} &= -\frac{\sin(\sqrt{x})\cos(\sqrt{x})}{\sqrt{x}} - \frac{\sin(\sqrt{x})\cos(\sqrt{x})}{\sqrt{x}} \\ &= -\frac{2\sin(\sqrt{x})\cos(\sqrt{x})}{\sqrt{x}} \\ &= \frac{-\sqrt{x} \sin(2\sqrt{x})}{x} \end{aligned}$$

2. The position of a particle, in metres, is described by $s(t) = 10\cos\left(5t - \frac{\pi}{4}\right)$, for $0 \leq t \leq \pi$, where t is time, in seconds.

- a) Determine the position of the particle when $t = 2$ seconds.

$$\begin{aligned} s(2) &= 10\cos\left[5(2) - \frac{\pi}{4}\right] \\ &\approx -9.78 \text{ m} \end{aligned}$$

$\therefore$ The particle is approx. 9.78m to the left of the reference point.

- b) Determine when the velocity is zero.

$$\begin{aligned} v(t) = s'(t) &= -10\sin\left(5t - \frac{\pi}{4}\right) \cdot (5) \\ &= -50\sin\left(5t - \frac{\pi}{4}\right) \end{aligned}$$

Find when $v(t) = 0$:

$$\begin{aligned} 0 &= -50\sin\left(5t - \frac{\pi}{4}\right) \\ 0 &= \sin\left(5t - \frac{\pi}{4}\right) \end{aligned}$$

$$\begin{aligned} \therefore 5t - \frac{\pi}{4} &= 0 & \text{or} & \quad 5t - \frac{\pi}{4} = \pi & \text{or} & \quad 5t - \frac{\pi}{4} = 2\pi & \text{or} & \quad 5t - \frac{\pi}{4} = 3\pi & \text{or} & \quad 5t - \frac{\pi}{4} = 4\pi \\ 5t &= \frac{\pi}{4} & & \quad 5t = \frac{5\pi}{4} & & \quad 5t = \frac{9\pi}{4} & & \quad 5t = \frac{13\pi}{4} & & \quad 5t = \frac{17\pi}{4} \\ t &= \frac{\pi}{20} & & \quad t = \frac{\pi}{4} & & \quad t = \frac{9\pi}{20} & & \quad t = \frac{13\pi}{20} & & \quad t = \frac{17\pi}{20} \end{aligned}$$

$\therefore$ Velocity is zero when $t = \frac{\pi}{20}, \frac{\pi}{4}, \frac{9\pi}{20}, \frac{13\pi}{20}$ and $\frac{17\pi}{20}$.

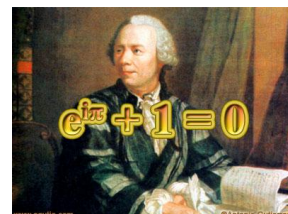
Alt solution using general solutions:

$$\begin{aligned} 5t - \frac{\pi}{4} &= k\pi \\ t &= \frac{k\pi + \frac{\pi}{4}}{5} \\ t &= \frac{k}{5}\pi + \frac{1}{20}\pi \\ &= \frac{(4k+1)\pi}{20}, \text{ sub } k=0,1,2,3,4 \end{aligned}$$

The Number e and the Derivatives of $y = e^x$ and $y = \ln x$

Introducing a Special Number, e

Leonhard Euler (1707-1783) was a remarkable Swiss mathematician and physicist. He made massive contributions to mathematics, especially calculus, as well as physics, optics, magnetism, astronomy, and shipbuilding. Euler popularized the use of the symbol π and developed new approximations for it. He was the first to use the symbol i to represent imaginary numbers. Euler also developed the irrational number e , which is known as Euler's number and is defined as a limit:



$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$$

Euler's Number e :

Let's examine some integer values of x to see the limiting value of this expression.

x	$\left(1 + \frac{1}{x} \right)^x$
1	2
10	2.5937
100	2.70481
1000	2.716923
10000	2.718145
100000	2.718268
1000000	2.718280

We see that as x becomes larger, the value of the expression changes by a smaller and smaller amount. In fact, the change can be shown to approach 0. In other words, the expression is approaching a limiting value. The limiting value of this expression is the **irrational** number $e = 2.718281828459\dots$, a non-terminating decimal.

By making the substitution $t = \frac{1}{x}$, we get the following alternate definition of Euler's number:

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = \lim_{t \rightarrow 0} (1 + t)^{\frac{1}{t}}$$

This limit is called the Fundamental Limit of Calculus.

The Inverse of the Exponential Function, e^x

In previous mathematics courses, you learned that the inverse of an exponential function is the logarithmic function with the same base. Today, we will use similar ideas to graph of $g(x) = e^x$ and $y = \log_e x$.

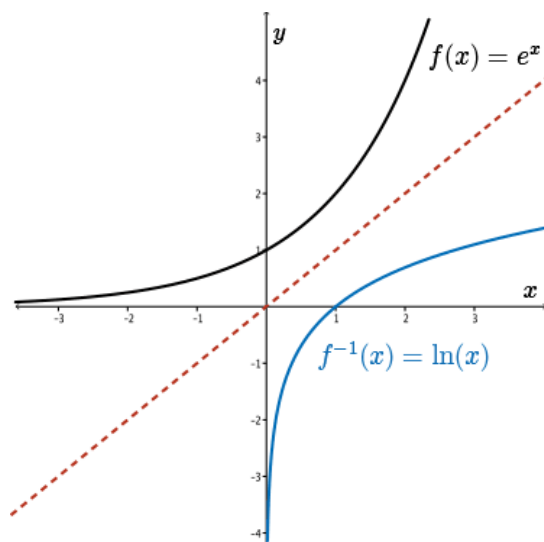
The inverse of the exponential function $g(x) = e^x$ is $g^{-1}(x) = \log_e(x) = \ln(x)$.

Rather than using $\log_e(x)$, mathematicians use **$\ln(x)$** to shorten this expression. **$\ln(x)$** stands for the natural logarithm of x and is pronounced “lawn x.”

Like every function, we can find its inverse by reflecting the function in the line $y = x$. Hence, to graph $y = \ln x$, we can interchange the values of x and y from the table of values of $y = e^x$.

x	$y = e^x$
-2	0.135
-1	0.36
0	1
1	2.72
2	7.39

x	$y = \ln x$
0.135	-2
0.36	-1
1	0
2.72	1
7.39	2



	$y = e^x$	$y = \ln x$
Domain	$\{x \in \mathbb{R}\}$	$\{x x > 0, x \in \mathbb{R}\}$ or $\{x \in \mathbb{R}, x > 0\}$
Range	$\{y y > 0, y \in \mathbb{R}\}$ or $\{y \in \mathbb{R} y > 0\}$	$\{y \in \mathbb{R}\}$

In advanced functions, you learned the **logarithmic laws and properties of logarithm**. We can apply similar laws and properties when working with $\ln$.

$\log_c ab = \log_c a + \log_c b$	$\ln(ab) = \ln(a) + \ln(b)$
$\log_c \frac{a}{b} = \log_c a - \log_c b$	$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
$\log_c a^n = n \log_c a$	$\ln(a^n) = n \ln(a)$
$\log_a a = 1$	$\ln e = 1$
$a^{\log_a x} = x$	$e^{\ln x} = x$
$\log_a 1 = 0$	$\ln 1 = 0$

Exponential form:

$$x = b^y$$

$\leftrightarrow$

Logarithmic form:

$$y = \log_b(x)$$

Example 1: Solve for x :

a) $5(10^{x+2}) = 200$

$$\begin{aligned} 10^{x+2} &= 40 \\ \log(10^{x+2}) &= \log(40) \\ x+2 &= \log(40) \\ x &= \log(40) - 2 \\ x &\doteq -0.4 \end{aligned}$$

b) $10e^{2x-1} + 500 = 1000$

$$\begin{aligned} 10e^{2x-1} &= 500 \\ e^{2x-1} &= 50 \\ \ln(e^{2x-1}) &= \ln(50) \\ 2x-1 &= \ln(50) \\ x &= \frac{\ln(50) - 1}{2} \\ x &\doteq 2.46 \end{aligned}$$

Example 2: Water is brought to a boil then removed from the heat. The temperature of the water, T degrees Celsius, is modeled as $T = 80e^{-0.57t} + 20$, where t is in minutes.

a) Determine the temperature after 15 minutes.

$$\begin{aligned} T &= 80e^{-0.57(15)} + 20 \\ &\doteq 20.0 \end{aligned}$$

$\therefore$ The temperature is approx. 20.0°C .

b) Determine how long it takes for the temperature to reach 30°C .

$$\begin{aligned} 30 &= 80e^{-0.57t} + 20 \\ \frac{1}{8} &= e^{-0.57t} \\ \ln\left(\frac{1}{8}\right) &= -0.57t \\ \frac{\ln\left(\frac{1}{8}\right)}{-0.57} &= t \\ 3.65 &\doteq t \end{aligned}$$

$\therefore$ It takes approx. 3.65 mins to reach 30°C .

Example 3: A population of fish in a lake at time t months is given by the function $F(t) = \frac{20000}{1 + 24e^{-\frac{t}{4}}}$.

How long will it take for the fish population to reach 15 000?

$$15000 = \frac{20000}{1 + 24e^{-\frac{t}{4}}}$$

$$1 + 24e^{-\frac{t}{4}} = \frac{4}{3}$$

$$e^{-\frac{t}{4}} = \frac{1}{72}$$

$$-\frac{t}{4} = \ln\left(\frac{1}{72}\right)$$

$$t = -4 \ln\left(\frac{1}{72}\right)$$

$$t \approx 17.1 \quad \therefore \text{It will take approx. 17.1 months.}$$

The Derivative of $y = e^x$ and $y = \ln x$

The Derivative of $f(x) = e^x$

Proof: Let $f(x) = e^x$.

$$\begin{aligned} \text{Then, from first principles: } f'(x) &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x(e^h - 1)}{h} \\ &= e^x \lim_{h \rightarrow 0} \frac{e^h - 1}{h} \\ &= e^x \end{aligned}$$

$$\text{Recall: } \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$$

$$\text{Let } x = e^h - 1 \quad \therefore x+1 = e^h$$

$$\ln(x+1) = h$$

$$\begin{aligned} \therefore \lim_{h \rightarrow 0} \frac{e^h - 1}{h} &= \lim_{x \rightarrow 0} \frac{x}{\ln(x+1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{1}{x} \ln(1+x)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\ln(1+x)^{\frac{1}{x}}} \\ &= \frac{1}{\lim_{x \rightarrow 0} [\ln(1+x)^{\frac{1}{x}}]} \\ &= \frac{1}{\ln \left[\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \right]} \\ &= \frac{1}{\ln(e)} \\ &= 1 \end{aligned}$$

If $f(x) = e^x$, then $f'(x) = e^x$.

By the chain rule, if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} \cdot g'(x)$.

$$\text{or } g'(x) e^{g(x)}$$

Examples: Differentiate.

a) $f(x) = 3e^x$

$$f'(x) = 3e^x$$

b) $f(x) = 2e^{x^2}$

$$f'(x) = 2e^{x^2} \cdot 2x = 4xe^{x^2}$$

c) $f(x) = x^3 e^{-x}$

$$f'(x) = 3x^2 e^{-x} + x^3 e^{-x} (-1) = x^2 e^{-x} (3 - x)$$

d) $f(x) = e^{x^2+3x}$

$$f'(x) = (2x+3)e^{x^2+3x}$$

The Derivative of $y = \ln x$

Proof: Let $e^y = x$

Differentiate both sides: $e^y \cdot \frac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

Sub $e^y = x$: $\frac{dy}{dx} = \frac{1}{x}$

If $y = \ln x$ then $y' = \frac{1}{x}$.

Derivative of $y = \ln g(x)$: $\frac{1}{g(x)} \cdot g'(x)$ or $\frac{g'(x)}{g(x)}$.

Ex 1: Differentiate and simplify.

a) $f(x) = x^2 \ln x$

$$f'(x) = 2x \ln(x) + x^2 \cdot \frac{1}{x} = 2x \ln(x) + x = x[2 \ln(x) + 1]$$

b) $f(x) = \frac{\ln x}{x}$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

c) $f(x) = \ln\left(\frac{x-1}{x+1}\right)$

$$f(x) = \ln(x-1) - \ln(x+1) \\ f'(x) = \frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1 - (x-1)}{x^2-1} = \frac{2}{x^2-1}$$

d) $f(x) = e^{x^2} \ln(\sqrt{x})$

$$f'(x) = e^{x^2} \cdot 2x \cdot \ln(\sqrt{x}) + e^{x^2} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = e^{x^2} \left[2x \ln(\sqrt{x}) + \frac{1}{2x} \right] = \frac{e^{x^2}}{2x} [4x^2 \ln(\sqrt{x}) + 1]$$

e) $f(x) = \ln(\sqrt{x^3+x^2})$

$$f'(x) = \frac{\frac{1}{2}(x^3+x^2)^{-\frac{1}{2}}(3x^2+2x)}{\sqrt{x^3+x^2}} = \frac{3x^2+2x}{2(x^3+x^2)} = \frac{3x+2}{2x(x+1)}$$

Ex 2: Find the equation of the tangent line to the curve $y = e^{-x}$ that passes through the origin.

$$y' = -e^{-x} \rightarrow \text{Need to find the point of tangency}$$

Slope of tangent using $(0,0), (x, e^{-x})$: $m_T = \frac{e^{-x} - 0}{x - 0} = \frac{e^{-x}}{x}$

$$y' = m_T \quad \therefore -e^{-x} = \frac{e^{-x}}{x}$$

$$-x e^{-x} = e^{-x}$$

$$e^{-x}(1+x) = 0$$

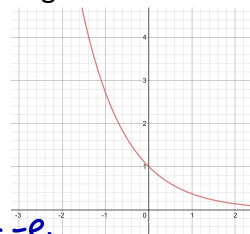
$$e^{-x} = 0 \quad \therefore 1+x = 0$$

DNE $x = -1$

$\therefore$ Point of tangency is $(-1, e)$ and $m_T = -e$

Equation is: $y - e = -e(x - (-1)) \rightarrow y = -ex$

$\therefore$ The equation of the tangent line is $y = -ex$.



Ex 3: Find the points of inflection for $y = f(x) = e^{-x^2}$.

$$f'(x) = -2x e^{-x^2}$$

$$f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2}$$

$$= -2e^{-x^2}(1 - 2x^2)$$

POI $\Rightarrow f''(x) = 0$

$$\therefore e^{-x^2} = 0 \quad \text{or} \quad 1 - 2x^2 = 0$$

DNE $x^2 = \frac{1}{2}$
 $x = \pm \frac{1}{\sqrt{2}}$

$$f''(x) \quad \begin{array}{c} + \quad - \quad + \\ \hline cu \quad -\frac{1}{\sqrt{2}} \quad cu \quad \frac{1}{\sqrt{2}} \quad cu \end{array}$$

$$f\left(\frac{1}{\sqrt{2}}\right) = e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = e^{-\frac{1}{2}}$$

$$f\left(-\frac{1}{\sqrt{2}}\right) = e^{-\left(-\frac{1}{\sqrt{2}}\right)^2} = e^{-\frac{1}{2}}$$

$\therefore$ POIs at $\left(-\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right), \left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$

Ex 4: Find local extrema points for $f(x) = \frac{\ln x}{x}, x > 0$

$$f'(x) = \frac{\frac{1}{x}(x) - \ln(x)}{x^2}$$

$$= \frac{1 - \ln(x)}{x^2}$$

$$f'(x) \quad \begin{array}{c} + \quad - \\ \hline 0 \quad \uparrow \quad e \quad \downarrow \end{array}$$

$$f(e) = \frac{\ln(e)}{e}$$

$$= \frac{1}{e}$$

$$f'(x) = 0 : 0 = 1 - \ln(x)$$

$$\ln(x) = 1$$

$$x = e$$

$\therefore$ There is a local max at $\left(e, \frac{1}{e}\right)$.

$$f'(x) \text{ DNE : } x^2 = 0$$

$$x = 0 \quad (\text{not in domain})$$

Ex 5: Find the inflection points for $f(x) = x^2 \ln x, x > 0$

$$f'(x) = 2x \ln(x) + x^2 \cdot \frac{1}{x}$$

$$= 2x \ln(x) + x$$

$$f''(x) \quad \begin{array}{c} - \quad + \\ \hline 0 \quad \downarrow \quad e^{-\frac{3}{2}} \quad \uparrow \quad cu \end{array}$$

$$f''(x) = 2 \ln(x) + 2x \cdot \frac{1}{x} + 1$$

$$= 2 \ln(x) + 3$$

$$f\left(e^{-\frac{3}{2}}\right) = \left(e^{-\frac{3}{2}}\right)^2 \ln\left(e^{-\frac{3}{2}}\right)$$

$$= -\frac{3}{2} e^{-3}$$

$$f''(x) = 0 : \ln(x) = -\frac{3}{2}$$

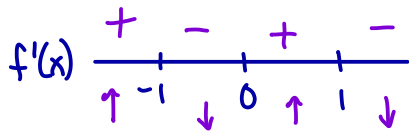
$$x = e^{-\frac{3}{2}}$$

$\therefore$ POI at $\left(e^{-\frac{3}{2}}, -\frac{3}{2}e^{-3}\right)$

Ex 6: Find the local extrema for $y = f(x) = x^2 e^{-x^2}$.

$$f'(x) = 2x e^{-x^2} + x^2 e^{-x^2} (-2x) \\ = 2x e^{-x^2} (1 - x^2)$$

$$f'(x) = 0 : x = 0 \text{ or } 1 - x^2 = 0 \\ x = \pm 1$$



Derivatives of $y = e^x$ and $y = \ln(x)$ Practice

$$f(-1) = (-1)^2 e^{-(-1)^2} \\ = e^{-1}$$

$$f(0) = 0$$

$$f(1) = (1)^2 e^{-(1)^2} \\ = e^{-1}$$

$\therefore$ local max at $(-1, \frac{1}{e})$ and $(1, \frac{1}{e})$
and local min at $(0, 0)$.

Multiple Choice: Identify the choice that best completes the statement or answers the question.

___ 1. The graph of $y = e^x$ lies between the graphs of which two functions?

- a. $y = 2x$ & $y = 3x$ b. $y = x^2$ & $y = x^3$ c. $y = 3^x$ & 4^x d. $y = 2^x$ & 3^x

___ 2. Determine the value of x in the equation $\ln x = 1$.

- a. 10 b. 0 c. 1 d. e

___ 3. Simplify the expression $\ln e^{3x}$.

- a. $3 \ln e^{3x}$ b. $\ln(3x)$ c. e^{3x} d. $3x$

___ 4. What is the value of $\ln 0$?

- a. 0 b. undefined c. 1 d. e

___ 5. Determine the value of x in the equation $2e^x = 6$.

- a. $\ln 6$ b. $\ln 3$ c. 3 d. $\frac{\ln 6}{\ln 3}$

___ 6. If $f(x) = 2 \sin x \cos x$, find $f'(\frac{\pi}{2})$.

- a. -2 b. 1 c. $\frac{\pi}{2} + 1$ d. 2

___ 7. If $f(x) = \cos(\sin x)$, find $f'(1)$.

- a. 0.2314 b. 1 c. 0 d. -0.4029

___ 8. What is the slope of the graph $y = 5e^x$ at $x = 1$?

- a. e b. $5e$ c. 5 d. $\ln 5$

___ 9. If $f(x) = 2x^2 e^x$, find $f'(1)$.

- a. $2e$ b. $6e$ c. 0 d. $4e$

- ____ 10. If $f(x) = e^{x^2-x+1}$, find $f'(1)$.
 a. 2 b. 1 c. 0 d. e
- ____ 11. If $f(x) = e^x \sqrt{x}$, find $f'(1)$.
 a. $\frac{e}{2}$ b. $2e$ c. $\frac{3e}{2}$ d. e
- ____ 12. If $f(x) = \sqrt{x} + \sqrt{e^x} + \sqrt{e}$, find $f'(1)$.
 a. $\frac{1 + \sqrt{e}}{2}$ b. $\frac{1}{\sqrt{e}}$ c. $1 + e$ d. $\frac{1}{2}$
- ____ 13. If $f(x) = \sin(e^x)$, find $f'(0)$.
 a. 1 b. 0 c. $\cos(1)$ d. $\cos(e)$
- ____ 14. Find the point on the curve $y = e^x + x$ where the tangent is parallel to the line $y = 2x$.
 a. $(1, 1 + e)$ b. $(0, 1)$ c. $(-1, -1 + \frac{1}{e})$ d. $(-2, -2 + \frac{1}{e^2})$
- ____ 15. Find the point on the curve $y = e^{2x+1}$ where the tangent is parallel to the line $y = 2ex$.
 a. $(1, e^3)$ c. $(\frac{1}{2}, e^2)$
 b. $(0, e)$ d. $(-1, \frac{1}{e})$

Full Solutions:

16. Find the derivative. You do NOT have to simplify:
 a. $f(x) = \ln(\tan^2(\sin x))$ b. $f(x) = e^{\sin x} \sec x$ c. $y = \ln(\ln 2x^4)$
17. Find the absolute maximum and absolute minimum of $f(x) = \frac{\ln x}{x}$ on the closed interval $x \in [2, 10]$.
18. For $f(x) = x^4 e^x$: Determine the intervals of increase and decrease.
19. Find y' (do not simplify): $y = (e^{\sqrt{x}} + x \sec x + 1)^3$
20. Find y'' if $y = x^2 e^{\sqrt{\sin^2 x + \cos^2 x + 1}}$ ☺
21. Simplify the expression $(e^{2 \ln x})(\ln e^{2x}) + \ln(e^{x+1})$
22. Solve for x (to nearest hundredth if necessary):
 a. $e^{-2x+3} = 4$ b. $\ln(x^2 - 2x) = 4$

Practice Questions 5.3: The Number e and the Functions $y = e^x$ and $y = \ln x$

Multiple Choice: Identify the choice that best completes the statement or answers the question.

- D.** 1. The graph of $y = e^x$ lies between the graphs of which two functions? **e is approximately 2.7**
 a. $y = 2x$ & $y = 3x$ b. $y = x^2$ & $y = x^3$ c. $y = 3^x$ & 4^x d. $y = 2^x$ & 3^x
- D.** 2. Determine the value of x in the equation $\ln x = 1$. **Using e as the base, represent it in exponential form.**
 a. 10 b. 0 c. 1 d. e
- D.** 3. Simplify the expression $\ln e^{3x}$. **Bring $3x$ in front and $\ln e = 1$.**
 a. $3 \ln e^{3x}$ b. $\ln(3x)$ c. e^{3x} d. $3x$
- B.** 4. What is the value of $\ln 0$? **0 is not part of the domain.**
 a. 0 b. undefined c. 1 d. e
- B.** 5. Determine the value of x in the equation $2e^x = 6$. **Divide both sides by 2 first.**
 a. $\ln 6$ b. $\ln 3$ c. 3 d. $\frac{\ln 6}{\ln 3}$

- A.** 6. If $f(x) = 2\sin x \cos x$, find $f'(\frac{\pi}{2})$.
 a. -2 b. 1 c. $\frac{\pi}{2} + 1$ d. 2

$$f(x) = \sin 2x$$

$$f'(x) = 2 \cos 2x$$

$$f'(\frac{\pi}{2}) = 2 \cos(\pi) = -2$$

- D.** 7. If $f(x) = \cos(\sin x)$, find $f'(1)$.
 a. 0.2314 b. 1 c. 0 d. -0.4029

$$f'(x) = -\sin(\sin x) * \cos x$$

$$f'(1) = \sin(\sin 1) * \cos 1 = -0.4029$$

- B.** 8. What is the slope of the graph $y = 5e^x$ at $x = 1$?
 a. E b. $5e$ c. 5 d. $\ln 5$

$$f'(x) = 5e^x$$

$$f'(1) = 5e^1$$

$$= 5e$$

- B.** 9. If $f(x) = 2x^2 e^x$, find $f'(1)$.
 a. $2e$ b. $6e$ c. 0 d. $4e$

$$f'(x) = 4x(e^x) + (2x^2)e^x$$

$$f'(1) = 4e + 2e = 6e$$

Practice Questions 5.3: The Number e and the Functions $y = e^x$ and $y = \ln x$

D 10. If $f(x) = e^{x^2-x+1}$, find $f'(1)$.

- a. 2 b. 1 c. 0 d. e

$$f'(x) = (2x - 1)e^{x^2-x+1}$$

$$f'(1) = e$$

C 11. If $f(x) = e^x \sqrt{x}$, find $f'(1)$.

- a. $\frac{e}{2}$ b. $2e$ c. $\frac{3e}{2}$ d. e

$$f'(x) = e^x \sqrt{x} + e^x \frac{1}{2\sqrt{x}}$$

$$f'(1) = e\sqrt{1} + e \frac{1}{2\sqrt{1}} = 2e$$

A 12. If $f(x) = \sqrt{x} + \sqrt{e^x} + \sqrt{e}$, find $f'(1)$.

- a. $\frac{1 + \sqrt{e}}{2}$ b. $\frac{1}{\sqrt{e}}$ c. $1 + e$ d. $\frac{1}{2}$

$$f'(x) = \sqrt{x} + e^{\frac{1}{2}x} + \sqrt{e}$$

$$f'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{2}e^{\frac{1}{2}x}$$

$$f'(1) = \frac{1 + \sqrt{e}}{2}$$

C 13. If $f(x) = \sin(e^x)$, find $f'(0)$.

- a. 1 b. 0 c. $\cos(1)$ d. $\cos(e)$

$$f'(x) = e^x \cos(e^x)$$

$$f'(0) = \cos 1$$

B 14. Find the point on the curve $y = e^x + x$ where the tangent is parallel to the line $y = 2x$.

- a. $(1, 1 + e)$ b. $(0, 1)$ c. $(-1, -1 + \frac{1}{e})$ d. $(-2, -2 + \frac{1}{e^2})$

$$y' = e^x + 1$$

$$\text{We need } y' = 2 \Rightarrow 2 = e^x + 1$$

$$1 = e^x$$

$$x = 0, y = 1$$

B 15. Find the point on the curve $y = e^{2x+1}$ where the tangent is parallel to the line $y = 2ex$.

- a. $(1, e^3)$ c. $(\frac{1}{2}, e^2)$
- b. $(0, e)$ d. $(-1, \frac{1}{e})$

$$y' = 2e^{2x+1}$$

$$\text{We need } y' = 2e \Rightarrow 2e = 2e^{2x+1}$$

$$1 = 2x + 1$$

$$x = 0, y = e$$

Practice Questions 5.3: The Number e and the Functions $y = e^x$ and $y = \ln x$

Full Solution:

16. Find the derivative. Simplify your answers:

$$\begin{aligned} \text{a. } f(x) &= \ln(\tan^2(\sin^x)) & \text{b. } f(x) &= e^{\sin x} \sec x & \text{c. } y &= \ln(\ln 2x^4) \\ y &= 2 \ln(\tan(\sin^x)) & y' &= e^{\sin x} (\cos x) \sec x + e^{\sin x} \sec x \tan x & y' &= \frac{1}{\ln 2x^4} \frac{1}{2x^4} (8x^3) \\ &= 2 \frac{1}{\tan(\sin^x)} \sec^2(\sin^x) (\cos^x) (e^x) & &= e^{\sin x} + e^{\sin x} \frac{\sin x}{\cos^2 x} & &= \frac{4}{x \ln 2x^4} \\ &= 2 \frac{\cos(\sin^x)}{\sin(\sin^x)} \frac{1}{\cos^2(\sin^x)} (\cos^x) (e^x) & & & & \\ &= \frac{2(\cos^x) e^x}{\sin(\sin^x) \cos(\sin^x)} = \frac{4(\cos^x) e^x}{\sin(2\sin^x)} \end{aligned}$$

17. Find the absolute maximum and absolute minimum of $f(x) = \frac{\ln x}{x}$ on the closed interval $x \in [2, 10]$.

$$y' = \frac{\frac{1}{x} - \ln x (1)}{x^2} = \frac{1 - \ln x}{x^2}$$

$$y' = 0 \rightarrow 1 - \ln x = 0 \rightarrow x = e.$$

Note: $x = 0$ is VA but is not part of the domain. Hence, $f(x)$ is continuous on the given domain.

$$f(2) \approx 0.35, f(10) \approx 0.23, f(e) \approx 0.37$$

$\therefore f(10)$ is the global maximum and $f(2)$ is global minimum.

18. For $f(x) = x^4 e^x$: Determine the intervals of increase and decrease

$$f'(x) = 4x^3 e^x + x^4 e^x = x^3 e^x (4 + x)$$

$$f'(x) = 0 \text{ implies } x = 0, -4$$

	$x < -4$	$-4 < x < 0$	$x > 0$
f'	+	-	+

$\therefore f(x)$ is increasing if $x \in (-\infty, -4) \cup (0, \infty)$ and decreasing if $x \in (-4, 0)$

19. Find y' (do not simplify): $y = (e^{\sqrt{x}} + x \sec x + 1)^3$

$$y' = 3 \left(e^{\sqrt{x}} + x \sec x + 1 \right)^2 \left(\frac{e^{\sqrt{x}}}{2\sqrt{x}} + \sec x + x \sec x \tan x \right)$$

Practice Questions 5.3: The Number e and the Functions $y = e^x$ and $y = \ln x$

20. Find y'' if $y = x^2 e^{\sqrt{\sin^2 x + \cos^2 x + 1}}$ ☺ Recall: $\sin^2 x + \cos^2 x = 1$

$$y = x^2 e^{\sqrt{2}}$$

$$y' = 2x e^{\sqrt{2}}$$

$$y'' = 2e^{\sqrt{2}}$$

21. Simplify the expression $(e^{2\ln x})(\ln e^{2x}) + \ln(e^{x+1})$

Using the properties of Natural Logarithm:

$$e^{2\ln x} = e^{\ln x^2} = x^2$$

$$\ln e^{2x} = 2x$$

$$\ln(e^{x+1}) = x + 1$$

$$\therefore (e^{2\ln x})(\ln e^{2x}) + \ln(e^{x+1}) = (x^2)(2x) + (x + 1) = 2x^3 + x + 1$$

22. Solve for x (to nearest hundredth if necessary):

a. $e^{-2x+3} = 4$

b. $\ln(x^2 - 2x) = 4$

Domain: $x^2 - 2x > 0$ which implies $x < 0$ or $x > 2$

$$\ln e^{-2x+3} = \ln 4$$

$$e^4 = x^2 - 2x$$

$$-2x + 3 = \ln 4$$

$$x^2 - 2x - e^4 = 0$$

$$x = \frac{\ln 4 - 3}{-2} \cong 0.81$$

Use Quadratic Formula: We get $x \cong 8.46, -6.46$

Warm Up: Derivative of Other Functions

1. Differentiate each of the following.

(a) $f(x) = \ln \sqrt{\frac{x-1}{x+1}} + \frac{\ln[\cos(x)]}{\sin(x)}$ (Do Not Simplify)

$$f'(x) = \sqrt{\frac{x+1}{x-1}} \cdot \frac{1}{2} \left(\frac{x-1}{x+1} \right)^{-\frac{1}{2}} \left[\frac{(x+1) - (x-1)}{(x+1)^2} \right] + \frac{\frac{-\sin(x)}{\cos(x)} \cdot \sin(x) - \ln[\cos(x)] \cdot \cos(x)}{\sin^2(x)}$$

(b) $y = e^{5\ln(x-3)} - e^2 \sin^3(e^x)$

$$y' = e^{5\ln(x-3)} \cdot \frac{5}{x-3} - 3e^2 \sin^2(e^x) \cdot e^x \cdot \cos(e^x)$$

2. Determine the exact point(s) of inflection of the function $f(x) = x(\ln(x))^2$

$$\begin{aligned} f'(x) &= [\ln(x)]^2 + x \cdot 2 \ln(x) \cdot \frac{1}{x} \\ &= [\ln(x)]^2 + 2 \ln(x) \end{aligned}$$

$$\begin{aligned} f''(x) &= 2 \ln(x) \cdot \frac{1}{x} + \frac{2}{x} \\ &= \frac{2 \ln(x) + 2}{x} \end{aligned}$$

Possible POIs : $f''(x) = 0$:

$$2 \ln(x) + 2 = 0$$

$$\ln(x) = -1$$

$$x = e^{-1}$$

$f''(x)$ DNE : $x=0 \leftarrow$ Not in domain

$$\begin{array}{c} \text{---} \quad \text{---} \quad \text{---} \\ f''(x) \quad 0 \quad \frac{1}{e} \quad \infty \\ \text{co} \quad \text{cu} \end{array}$$

$$\begin{aligned} f(e^{-1}) &= (e^{-1}) [\ln(e^{-1})]^2 \\ &= \frac{1}{e} \end{aligned}$$

$\therefore$ POI at $(\frac{1}{e}, \frac{1}{e})$

The Derivative of Exponential Functions $f(x) = b^x$

Proof: Let $f(x) = b^x$

$$f'(x) = \lim_{h \rightarrow 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h \rightarrow 0} \frac{b^x(b^h - 1)}{h}$$

$$= b^x \underbrace{\lim_{h \rightarrow 0} \frac{b^h - 1}{h}}_{\ln(b)}$$

$$= b^x \ln(b)$$

Let $x = a^h - 1 \therefore x + 1 = a^h$

$$\ln(x+1) = h \ln(a)$$

As $h \rightarrow 0$, $a^h - 1 \rightarrow 0 \therefore x \rightarrow 0$

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \lim_{x \rightarrow 0} \frac{x \ln(a)}{\ln(1+x)}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(a)}{\frac{1}{x} \ln(1+x)}$$

$$= \lim_{x \rightarrow 0} \frac{\ln(a)}{\ln(1+x)^{\frac{1}{x}}}$$

$$= \frac{\ln(a)}{\ln \left[\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \right]}$$

$$= \frac{\ln(a)}{\ln(e)} = \ln(a)$$

The derivative of $y = b^x$ is: $y' = b^x \cdot \ln(b)$.

The derivative of $y = b^{g(x)}$ is: $y' = b^{g(x)} \cdot g'(x) \cdot \ln(b)$.

Ex 1: Differentiate.

a) $f(x) = 3^x$

$$f'(x) = 3^x \cdot \ln(3)$$

b) $f(x) = \left(\frac{1}{2}\right)^x$

$$f'(x) = \left(\frac{1}{2}\right)^x \cdot \ln\left(\frac{1}{2}\right)$$

$$= -2^{-x} \ln 2$$

c) $f(x) = 3^{x^2}$

$$f'(x) = 2x \cdot 3^{x^2} \cdot \ln(3)$$

d) $f(x) = 4^{x^3+2x} \cdot 3^x$

$$f'(x) = 4^{x^3+2x} \cdot (3x^2+2) \cdot \ln(4) \cdot 3^x + 4^{x^3+2x} \cdot 3^x \cdot \ln(3)$$

$$= 3^x \cdot 4^{x^3+2x} \left[\ln(4) \cdot (3x^2+2) + \ln(3) \right]$$

Ex 2: Find the equation of the tangent line to the graph of $y = f(x) = x(2^{-x})$ at $(0, 0)$.

$$f'(x) = 2^{-x} + x(2^{-x}) \cdot \ln(2) \cdot (-1)$$

$$= 2^{-x} - x(2^{-x}) \ln(2)$$

$\therefore$ The equation is $y = x$.

$$f'(0) = 2^{-(0)} - 0$$

$$= 1$$

The Derivative of Logarithmic Functions $f(x) = \log_b(x)$

Proof: Let $y = \log_b(x)$

Exponential form: $x = b^y$

Differentiate: $1 = b^y \cdot \ln(b) \cdot \frac{dy}{dx}$

$$\frac{1}{b^y \cdot \ln(b)} = \frac{dy}{dx}$$

$$\therefore x = b^y : \therefore \frac{dy}{dx} = \frac{1}{x \ln(b)}$$

The derivative of $y = \log_b(x)$ is: $y' = \frac{1}{x \ln(b)}$.

The derivative of $y = \log_b g(x)$ is: $y' = \frac{g'(x)}{g(x) \cdot \ln(b)}$.

Ex 1: Differentiate.

a) $f(x) = x^2 \log_3 x$

$$f'(x) = 2x \log_3(x) + x^2 \cdot \frac{1}{x \ln(3)}$$

$$= x \left[2 \log_3(x) + \frac{1}{\ln(3)} \right]$$

b) $f(x) = \frac{\log x}{10^x}$

$$f'(x) = \frac{\frac{1}{x \ln(10)} \cdot 10^x - \log(x) \cdot 10^x \cdot \ln(10)}{10^{2x}}$$

$$= \frac{10^x \left[\frac{1}{x \ln(10)} - \ln(10) \log(x) \right]}{10^{2x}}$$

$$= \frac{1 - x [\ln(10)]^2 \log(x)}{x \cdot \ln(10) \cdot 10^x}$$

Ex 2: Differentiate.

a) $y = \log \sqrt{x^2 + 1}$

$$y' = \frac{\left(\frac{1}{2}\right)(x^2 + 1)^{-\frac{1}{2}} \cdot (2x)}{\sqrt{x^2 + 1} \cdot \ln(10)}$$

$$= \frac{x}{(x^2 + 1) \ln(10)}$$

c) $f(x) = \log(\ln x)$

$$f'(x) = \frac{\frac{1}{x}}{\ln(x) \cdot \ln(10)}$$

$$= \frac{1}{x \ln(x) \ln(10)}$$

b) $f(x) = \log_2(x^2 2^x)$

$$f'(x) = \frac{2x \cdot 2^x + x^2 \cdot 2^x \cdot \ln(2)}{x^2 \cdot 2^x \cdot \ln(2)}$$

$$= \frac{2}{x \ln(2)} + 1$$

Ex 3: Find the absolute extrema for $f(x) = \frac{\log x}{x}$ over the interval $[1, 10]$.

$$f'(x) = \frac{\frac{1}{x \ln(10)} \cdot x - \log(x)}{x^2}$$

$$= \frac{1 - \ln(10) \log(x)}{\ln(10) \cdot x^2}$$

$$f'(x) = 0: 1 - \ln(10) \log(x) = 0$$

$$\log(x) = \frac{1}{\ln(10)}$$

$$x = 10^{\frac{1}{\ln(10)}}$$

$$x \approx 2.72$$

$$f(1) = \frac{\log(1)}{(1)} = 0$$

$$f(10^{\frac{1}{\ln(10)}}) = \frac{\log(10^{\frac{1}{\ln(10)}})}{(10^{\frac{1}{\ln(10)}})} \approx 0.160$$

$$f(10) = \frac{\log(10)}{(10)}$$

$$= \frac{1}{10}$$

$\therefore$ The absolute min is at $(1, 0)$ and the absolute max is at approx. $(2.72, 0.160)$.

Ex 4. Determine the point on the graph of $f(x) = \ln(x+4+e^{-3x})$ where the tangent line is horizontal.

$$f'(x) = \frac{1 + e^{-3x} \cdot (-3)}{x+4+e^{-3x}}$$

$$= \frac{1-3e^{-3x}}{x+4+e^{-3x}}$$

horizontal $\rightarrow f'(x) = 0$

$$1-3e^{-3x} = 0$$

$$e^{-3x} = \frac{1}{3}$$

$$-3x = \ln\left(\frac{1}{3}\right)$$

$$x = -\frac{1}{3} \ln\left(\frac{1}{3}\right)$$

$$\doteq 0.366$$

$$f(0.366) \doteq 1.55$$

$\therefore$ The point on the graph where the tangent line is horizontal is approx. at $(0.366, 1.55)$.

Ex 5: State the intervals of concavity for the function $f(x) = x \ln(4-x^2)$.

$\rightarrow$ restriction: $4-x^2 > 0$
 $-x^2 > -4$
 $x^2 < 4$
 $-2 < x < 2$

$$f'(x) = \ln(4-x^2) + \frac{x(-2x)}{4-x^2}$$

$$= \ln(4-x^2) - \frac{2x^2}{4-x^2}$$

$$f''(x) = \frac{-2x}{4-x^2} - \left[\frac{4x(4-x^2) - 2x^2(-2x)}{(4-x^2)^2} \right]$$

$$= \frac{-2x(4-x^2) - 4x(4-x^2) - 4x^3}{(4-x^2)^2}$$

$$= \frac{-8x + 2x^3 - 16x + 4x^3 - 4x^3}{(4-x^2)^2}$$

$$= \frac{2x^3 - 24x}{(4-x^2)^2}$$

$$f''(x) = 0: 2x(x^2 - 12) = 0$$

$$\therefore x = 0$$

$$x^2 - 12 = 0$$

$$x = \pm 2\sqrt{3}$$

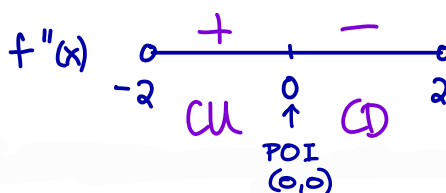
$\leftarrow$ inadmissible
 $\because -2 < x < 2$

$$f''(x) \text{ DNE} = (4-x^2)^2 = 0$$

$$4-x^2 = 0$$

$$x = \pm 2$$

$\leftarrow$ inadmissible
 $\because -2 < x < 2$



$\therefore f(x)$ is concave up on the interval $(-2, 0)$
 and concave down on $(0, 2)$.

The Derivative of Exponential and Logarithmic Functions Practice

Multiple Choice:

Identify the choice that best completes the statement or answers the question.

1. If $f(x) = 2^{\sin x}$, find $f'(0)$
 - a. 2
 - b. π
 - c. $\ln 2$
 - d. 1
2. If $f(x) = 90(2^{\frac{x}{3}})$, find $f'(3)$
 - a. $60\ln 2$
 - b. 30
 - c. $90\ln 2$
 - d. $180\ln 2$
3. If $f(x) = 2(3^{\cos x})$, find $f'(\frac{\pi}{2})$.
 - a. -2
 - b. 6
 - c. $-2\ln 3$
 - d. 2π

Full Solutions:

4. Find the derivative:
 - a. $y = (10^{\sin x})3^x$
 - b. $y = \sec^5(10^{e^x})$
 - c. $y = \log_5 \frac{x^2+1}{x-1}$
 - d. $y = \log_7 \sqrt{(x-1)(2x+1)(x+5)}$
 - e. $y = \sec(\frac{x}{5^x})\cos^2(\frac{x}{5^x})$
 - f. $y = 5^x \log e^{3x}$
5. Water is brought to a boil then removed from the heat. The temperature of the water, T degrees Celcius is modeled as $T = 80e^{-0.57t} + 20$. At what rate is the temperature decreasing when the temperature reaches 30°C .
6. The velocity of a car is given by $v(t) = 60[1 - (0.7)^t]$, where v is measured in m/s and t is measured in seconds. Determine the time at which the acceleration is 3 m/s^2 .
7. The mass of polonium is given by the function, $M(t) = M_0 \left(\frac{1}{2}\right)^{\frac{t}{138}}$ where M_0 is the initial mass of polonium, in milligrams, and $M(t)$ is the mass, in milligrams, after t days. At what rate is the polonium decaying when a 100 mg sample has decayed to 75% of its initial mass?

5.7: The Derivative of Exponential Functions-Solutions

Multiple Choice

Identify the choice that best completes the statement or answers the question.

1. If $f(x) = 2^{\sin x}$, find $f'(0)$

- a. 2 b. π c. $\ln 2$ d. 1

$$f'(x) = 2^{\sin x}(\cos x)(\ln 2)$$

$$f'(0) = 2^{\sin 0} \cos 0 (\ln 2) = \ln 2$$

2. If $f(x) = 90(2^{\frac{x}{3}})$, find $f'(3)$

- a. $60\ln 2$ b. 30 c. $90\ln 2$ d. $180\ln 2$

$$f'(x) = 90 \left(2^{\frac{x}{3}} \right) (\ln 2) \left(\frac{1}{3} \right)$$

$$f'(3) = 90(2) (\ln 2) \left(\frac{1}{3} \right) = 60\ln 2$$

3. If $f(x) = 2(3^{\cos x})$, find $f'(\frac{\pi}{2})$.

- a. -2 b. 6 c. $-2\ln 3$ d. 2π

$$f'(x) = 2(3^{\cos x})(\ln 3)(-\sin x)$$

$$f'(\frac{\pi}{2}) = 2 \left(3^{\cos \frac{\pi}{2}} \right) (\ln 3) \left(-\sin \frac{\pi}{2} \right) = -2\ln 3$$

4. Find the derivative:

a. $y = (10^{\sin x})3^x$

$$y' = (10^{\sin x})(\ln 10)(\cos x)3^x + (10^{\sin x})3^x(\ln 3)$$

$$= (10^{\sin x})3^x[\ln 10 \cos x + \ln 3]$$

b. $y = \sec^5(10^{e^x})$

$$y' = 5\sec^4(10^{e^x}) \sec(10^{e^x}) \tan(10^{e^x}) 10^{e^x} (\ln 10)(e^x)$$

$$= \frac{5 \sin(10^{e^x}) 10^{e^x} (\ln 10)(e^x)}{\cos^5(10^{e^x})}$$

d. $y = \log_7 \sqrt{(x-1)(2x+1)(x+5)}$

$$y = \frac{1}{2} \log_7 (x-1)(2x+1)(x+5)$$

$$= \frac{1}{2} [\log_7 (x-1) + \log_7 (2x+1) + \log_7 (x+5)]$$

$$y' = \frac{1}{2(x-1)\ln 7} + \frac{2}{2(2x+1)\ln 7} + \frac{1}{2(x+5)\ln 7}$$

$$= \frac{1}{2(x-1)\ln 7} + \frac{1}{(2x+1)\ln 7} + \frac{1}{2(x+5)\ln 7}$$

e. $y = \sec(\frac{x}{5^x}) \cos^2(\frac{x}{5^x})$

$$= \cos(\frac{x}{5^x})$$

$$y' = -\sin(\frac{x}{5^x}) \frac{5^x - x5^x \ln 5}{5^{2x}} \quad \text{Note: factor } 5^x \text{ and simplify}$$

$$= \frac{-\sin(\frac{x}{5^x})(1-x\ln 5)}{5^x}$$

$$c. \ y = \log_5 \frac{x^2+1}{x-1}$$

$$y = \log_5(x^2 + 1) - \log_5(x - 1)$$

$$y' = \frac{2x}{(x^2+1)\ln 5} - \frac{1}{(x-1)\ln 5}$$

$$f. \ y = 5^x \log e^{3x}$$

$$y = 3x(5^x) \log e$$

$$y' = \log e[3(5^x) + 2x(5^x)(\ln 5)]$$

$$= \log e(5^x)(3 + 2x(\ln 5))$$

5. Water is brought to a boil then removed from the heat. The temperature of the water, T degrees Celcius is modeled as $T = 80e^{-0.57t} + 20$. At what rate is the temperature decreasing when the temperature reaches 30°C .

We need $T'(t)$ when $T(t) = 30$.

$$30 = 80e^{-0.57t} + 20$$

$$10 = 80e^{-0.57t}$$

$$\frac{1}{8} = e^{-0.57t}$$

$$t = \frac{\ln(\frac{1}{8})}{-0.57}$$

$$T'(t) = 80(-0.57)e^{-0.57t}$$

$$T'\left(\frac{\ln(\frac{1}{8})}{-0.57}\right) = 80(-0.57)e^{-0.57\frac{\ln(\frac{1}{8})}{-0.57}} = 80(-0.57)\left(\frac{1}{8}\right) = -5.7$$

6. The velocity of a car is given by $v(t) = 60[1 - (0.7)^t]$, where v is measured in m/s and t is measured in seconds. Determine the time at which the acceleration is 3 m/s^2 .

$$v(t) = 60[1 - (0.7)^t]$$

$$a(t) = 60[-(0.7)^t \ln(0.7)]$$

$$3 = 60[-(0.7)^t \ln(0.7)]$$

$$\frac{-1}{20 \ln(0.7)} = (0.7)^t$$

$$0.14 = (0.7)^t$$

$$t = 5.5 \text{ sec}$$

7. The mass of polonium is given by the function, $M(t) = M_0 \left(\frac{1}{2}\right)^{\frac{t}{138}}$ where M_0 is the initial mass of polonium, in milligrams, and $M(t)$ is the mass, in milligrams, after t days. At what rate is the polonium decaying when a 100 mg sample has decayed to 75% of its initial mass?

$$\frac{75}{100} M_0 = M_0 \left(\frac{1}{2}\right)^{\frac{t}{138}}$$

$$\frac{3}{4} = \left(\frac{1}{2}\right)^{\frac{t}{138}}$$

$$\ln\left(\frac{3}{4}\right) = \frac{t}{138} \ln\left(\frac{1}{2}\right) \Rightarrow t = 57.3 \text{ days}$$

$$M'(57.3) = ?$$

$$M'(t) = \frac{100}{138} \left(\frac{1}{2}\right)^{\frac{t}{138}} \ln\left(\frac{1}{2}\right)$$

$$\begin{aligned} M'(57.3) &= \left(\frac{100}{138}\right) \left(\frac{1}{2}\right)^{\frac{57.3}{138}} \ln\left(\frac{1}{2}\right) \\ &= -0.38 \text{ mg/days} \end{aligned}$$

The mass decreases at a rate of 0.38 milligram per day.

5-9 Warm Up

Differentiate:

$$\textcircled{1} y = e^{\sqrt{x}}$$

$$\textcircled{2} f(x) = x e^{-x^2}$$

$$\textcircled{3} y = 3x + 2e^x$$

$$\textcircled{4} y = \frac{1}{\sqrt{2e^{-x} + 1}}$$

$$\textcircled{5} y = 3^x e^x$$

$$\textcircled{6} y = e^{-3x^2}$$

$$\textcircled{7} y = e^{\cos(x)}$$

$$\textcircled{8} y = e^{4x} \sin(2x)$$

Solutions:

$$\begin{aligned} \textcircled{1} y' &= e^{\sqrt{x}} \cdot \frac{1}{2} x^{-\frac{1}{2}} \\ &= \frac{e^{\sqrt{x}}}{2\sqrt{x}} \end{aligned}$$

$$\begin{aligned} \textcircled{2} f'(x) &= e^{-x^2} + x \cdot e^{-x^2} (-2x) \\ &= e^{-x^2} - 2x^2 e^{-x^2} \\ &= e^{-x^2} (1 - 2x^2) \\ &= \frac{1 - 2x^2}{e^{x^2}} \end{aligned}$$

$$\textcircled{3} y' = 3 + 2e^x$$

$$\begin{aligned} \textcircled{4} y &= (2e^{-x} + 1)^{-\frac{1}{2}} \\ \frac{dy}{dx} &= -\frac{1}{2} (2e^{-x} + 1)^{-\frac{3}{2}} (2e^{-x}) (-1) \\ &= \frac{1}{e^x (2e^{-x} + 1)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \textcircled{5} y' &= 3^x \cdot \ln 3 \cdot e^x + 3^x \cdot e^x \\ &= 3^x e^x [\ln(3) + 1] \end{aligned}$$

$$\begin{aligned} \textcircled{6} y' &= e^{-3x^2} (-6x) \\ &= \frac{-6x}{e^{3x^2}} \end{aligned}$$

$$\begin{aligned} \textcircled{7} y' &= e^{\cos x} (-\sin x) \\ &= -e^{\cos x} \sin x \end{aligned}$$

$$\begin{aligned} \textcircled{8} y' &= e^{4x} \cdot 4 \cdot \sin(2x) + e^{4x} \cdot \cos(2x) \cdot 2 \\ &= 2e^{4x} [2\sin(2x) + \cos(2x)] \end{aligned}$$

Optimization Problems with Exponential Functions

- Ex. 1: The effectiveness of studying for a test depends on how many hours a student studies. Some experiments showed that if the effectiveness, E , is put on a scale of 0 to 10, then $E(t) = 0.5 \left(10 + t e^{-\frac{t}{20}} \right)$, where t is the number of hours spent studying for an examination. If a student has up to 30 h that he can spend studying, how many hours should he study for maximum effectiveness?

Domain: $0 \leq t \leq 30$

$$E'(t) = 0.5 \left[e^{-\frac{t}{20}} + t e^{-\frac{t}{20}} \cdot \left(-\frac{1}{20}\right) \right]$$

$$= 0.5 e^{-\frac{t}{20}} \left(1 - \frac{t}{20} \right)$$

$$E'(t) = 0 : e^{-\frac{t}{20}} = 0 \quad \text{or} \quad 1 - \frac{t}{20} = 0$$

no sol'ns

$$\frac{t}{20} = 1$$

$$t = 20$$

Use endpoints to check for maximum effectiveness (absolute max):

$$E(0) = 0.5 \left[10 + (0) e^{\frac{0}{20}} \right] = 5$$

$$E(20) = 0.5 \left[10 + (20) e^{-\frac{20}{20}} \right] \doteq 8.68$$

$$E(30) = 0.5 \left[10 + (30) e^{-\frac{30}{20}} \right] \doteq 8.35$$

$\therefore$ The maximum effectiveness is achieved when a student studies 20h for a test.

- Ex 2: The number of insects on a small tropical island is given by $P(t) = 8000e^{0.03t}$ for the period starting January 1, 2001. $P(t)$ represents the insect population after t months.

- a) What was the population on January 1, 2001?

$$P(0) = 8000 e^{0.03(0)}$$

$$= 8000$$

$\therefore$ The initial population was 8000.

- b) When would the population double? Round to 1 decimal place. (exact date)

$$16000 = 8000 e^{0.03t}$$

$$2 = e^{0.03t}$$

$$\ln(2) = 0.03t$$

$$\frac{\ln(2)}{0.03} = t \rightarrow t \doteq 23.1$$

(23 months and 3 days)

$\therefore$ The population will double on Dec 3rd, 2002.

- c) Determine the rate of growth in population after 4 months.

$$P'(t) = 8000 e^{0.03t} \cdot (0.03)$$

$$= 240 e^{0.03t}$$

$$P'(4) = 240 e^{0.03(4)}$$

$$\doteq 271$$

$\therefore$ The rate of growth is approx. 271 insects per month.

Logarithmic Differentiation

Steps:

1. Take natural logarithms of both sides of an equation $y = f(x)$.
2. Simplify using logarithmic properties and laws.
3. Differentiate implicitly with respect to x .
4. Solve the resulting equation for y' or $\frac{dy}{dx}$. Note: you may need to substitute the expression for y into the derivative to express the derivative entirely in terms of x .

Ex. 1. Differentiate: $y = \frac{e^x \sqrt{x^2 - 1}}{(x+3)^2}$

Take the $\ln$
of both sides

$$\ln y = \ln \left[\frac{e^x \sqrt{x^2 - 1}}{(x+3)^2} \right]$$

$$\ln y = \ln e^x + \ln (x^2 - 1)^{\frac{1}{2}} - \ln (x+3)^2$$

$$\ln y = x + \frac{1}{2} \ln (x^2 - 1) - 2 \ln (x+3)$$

Differentiate

$$\frac{1}{y} \frac{dy}{dx} = 1 + \frac{1}{2} \left(\frac{2x}{x^2 - 1} \right) - \frac{2}{x+3}$$

$$\frac{dy}{dx} = y \left[1 + \frac{x}{x^2 - 1} - \frac{2}{x+3} \right]$$

sub expression
for y

$$\frac{dy}{dx} = \frac{e^x \sqrt{x^2 - 1}}{(x+3)^2} \left[1 + \frac{x}{x^2 - 1} - \frac{2}{x+3} \right]$$

Ex. 2. Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{x^2+1}{x^2+2}}$, $y > 0$ for all $x \in \mathbb{R}$

$$\ln y = \ln \left(\frac{x^2+1}{x^2+2} \right)^{\frac{1}{2}}$$

$$\ln y = \frac{1}{2} [\ln(x^2+1) - \ln(x^2+2)]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{2} \left[\frac{2x}{x^2+1} - \frac{2x}{x^2+2} \right]$$

$$\frac{dy}{dx} = y \left[\frac{x}{x^2+1} - \frac{x}{x^2+2} \right]$$

$$\frac{dy}{dx} = \sqrt{\frac{x^2+1}{x^2+2}} \left[\frac{x}{x^2+1} - \frac{x}{x^2+2} \right]$$

$$\frac{dy}{dx} = x \sqrt{\frac{x^2+1}{x^2+2}} \left(\frac{1}{x^2+1} - \frac{1}{x^2+2} \right)$$

Ex. 3. Differentiate: $y = (\sin x)^x$, $\sin x > 0$.

$$\ln y = x \ln(\sin x)$$

$$\frac{1}{y} \frac{dy}{dx} = \ln(\sin x) + x \left(\frac{1}{\sin x} \right) \cdot \cos x$$

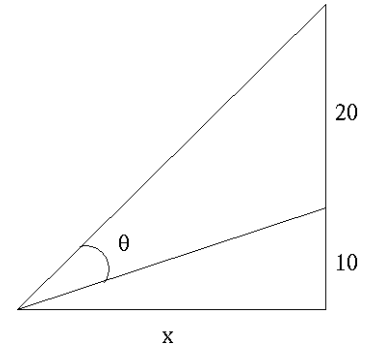
$$\frac{dy}{dx} = y [\ln \sin x + x \cot x]$$

$$\frac{dy}{dx} = (\sin x)^x [\ln(\sin x) + x \cot x]$$

Unit 5: Optimization for Exponential Functions-HW

1. Suppose that the monthly revenue in thousands of dollars, for the sale of x hundred units of an electronic item is given by the function $R(x) = 40x^2 e^{-0.4x} + 30$, where the maximum capacity of the plant is 800 units. Determine the number of units to produce in order to maximize revenue.
2. The hypotenuse of a right triangle is 12 cm. Calculate the measure of the unknown angles in the triangle that will maximize the perimeter.

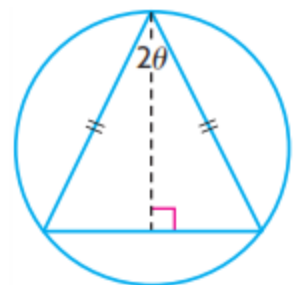
- Don't do! X 3. A movie screen on a wall is 20 meter high and 10 meter above the floor. At what distance x from the front of the room should you position yourself so that the tan ratio of viewing angle θ of the movie screen is as large as possible. (Hint: Maximize $\tan(\theta)$).



4. A drug is injected into the body in such a way that the concentration, C , in the blood at time t hours is given by the function $C(t) = 10(e^{-2t} - e^{-3t})$. At what time does the highest concentration occur within the first 5 h?

- Don't do! X 5. Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where the wire should stake to have the **angle** formed by the two pieces of wire at the stake be maximum?

6. An isosceles triangle is inscribed in a circle of radius R . Find the value of θ that maximizes the area of the triangle.



Unit 5: Optimization Problems-Solutions

- Suppose that the monthly revenue in thousands of dollars, for the sale of x hundred units of an electronic item is given by the function $R(x) = 40x^2e^{-0.4x} + 30$, where the maximum capacity of the plant is 800 units. Determine the number of units to produce in order to maximize revenue.

$$R'(x) = 40 \left[2xe^{-0.4x} - 0.4x^2e^{-0.4x} \right]$$

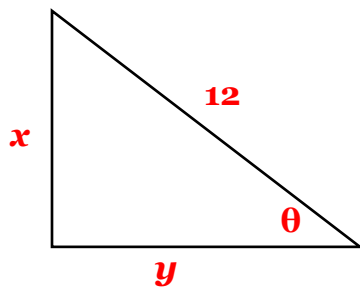
$$= 40xe^{-0.4x} (2 - 0.4x)$$

$$R'(x) = 0 : \boxed{x = 0}, 2 - 0.4x = 0$$

$$\boxed{x = 5}$$

$\therefore$ 500 units must be produced

- The hypotenuse of a right triangle is 12 cm. Calculate the measure of the unknown angles in the triangle that will maximize the perimeter.



$$P = x + y + 12$$

$$P(\theta) = 12\sin(\theta) + 12\cos(\theta) + 12 ; 0 < \theta < \frac{\pi}{2}$$

$$P'(\theta) = 12\cos(\theta) - 12\sin(\theta)$$

$$0 = 12(\cos(\theta) - \sin(\theta))$$

$$\cos(\theta) - \sin(\theta) = 0$$

$$\sin(\theta) = \cos(\theta)$$

$$\tan(\theta) = 1$$

$$\boxed{\theta = \frac{\pi}{4}}$$

3. A movie screen on a wall is 20 meter high and 10 meter above the floor. At what distance x from the front of the room should you position yourself so that the tan ratio of viewing angle θ of the movie screen is as large as possible. (maximize $\tan\theta$).

$$\tan(\alpha) = \frac{10}{x}$$

$$\tan(\alpha + \theta) = \frac{30}{x}$$

$$\frac{\tan(\alpha) + \tan(\theta)}{1 - \tan(\alpha)\tan(\theta)} = \frac{30}{x}$$

$$\frac{\frac{10}{x} + \tan(\theta)}{1 - \left(\frac{10}{x}\right)\tan(\theta)} = \frac{30}{x}$$

$$\frac{10 + x\tan(\theta)}{x - 10\tan(\theta)} = \frac{30}{x}$$

$$10x + x^2\tan(\theta) = 30x - 300\tan(\theta)$$

$$\tan(\theta)(x^2 + 300) = 20x$$

$$\tan(\theta) = \frac{20x}{x^2 + 300}$$

$$f(x) = \frac{20x}{x^2 + 300}$$

$$f'(x) = \frac{20(x^2 + 300) - 40x^2}{(x^2 + 300)^2}$$

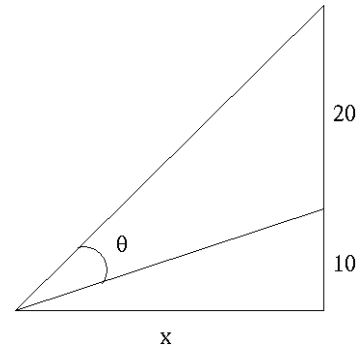
$$f'(x) = 0 : 600 - 20x^2 = 0$$

$$30 = x^2$$

$$\sqrt{30} = x$$

$$\tan(\theta) = \sqrt{30}$$

$$\theta \approx 79.6^\circ$$



4. A drug is injected into the body in such a way that the concentration, C , in the blood at time t hours is given by the function $C(t) = 10(e^{-2t} - e^{-3t})$. At what time does the highest concentration occur within the first 5 h?

$$C'(t) = 10(-2e^{-2t} + 3e^{-3t}); 0 \leq t \leq 5$$

$$C'(t) = 0 : -2e^{-2t} + 3e^{-3t} = 0$$

$$e^{-3t}(-2e^t + 3) = 0$$

$$e^t = \frac{3}{2}$$

$$t = \ln\left(\frac{3}{2}\right)$$

$$t \approx 0.41 \text{ hr}$$

- ✕ 5. Two poles, one 6 meters tall and one 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the **angle** formed by the two pieces of wire at the stake is a maximum?

$$L = l_1 + l_2$$

$$\tan(\alpha) = \frac{6}{x}$$

$$\tan(\beta) = \frac{15}{20-x}$$

$$\theta = \pi - (\alpha + \beta)$$

$$\tan(\theta) = \tan(\pi - (\alpha + \beta))$$

$$\tan(\theta) = -\tan(\alpha + \beta)$$

$$= -\frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$= -\frac{\frac{6}{x} + \frac{15}{20-x}}{1 - \left(\frac{6}{x}\right)\left(\frac{15}{20-x}\right)}$$

$$= -\frac{120 - 6x + 15x}{20x - x^2 - 90}$$

$$= \frac{9x + 120}{x^2 - 20x + 90}$$

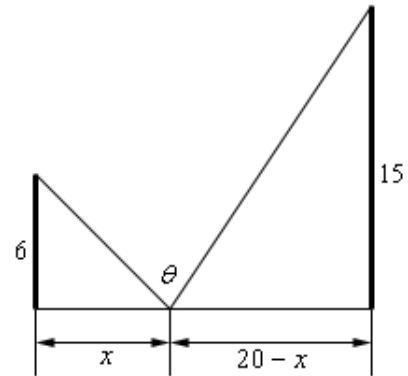
$$\text{Let } f(x) = \frac{9x + 120}{x^2 - 20x + 90}, \text{ then } f'(x) = 0$$

$$9(x^2 - 20x + 90) - (2x - 20)(9x + 120) = 0$$

$$9x^2 + 240x - 3210 = 0$$

$$x = 9.7847 \text{ or } x = -36.4514$$

$$\theta = 92.7^\circ$$



6. An isosceles triangle is inscribed in a circle of radius R . Find the value of θ that maximizes the area of the triangle.

$$\sin(2\theta) = \frac{y}{R} \quad \uparrow y = R \sin(2\theta)$$

$$\cos(2\theta) = \frac{h}{R} \quad \uparrow h = R \cos(2\theta)$$

$$A = (R + h)y$$

$$A = (R + R \cos(2\theta))(R \sin(2\theta))$$

$$A(\theta) = R^2 (\sin(2\theta) + \cos(2\theta) \sin(2\theta)); \quad 0 \leq 2\theta \leq \frac{\pi}{2}$$

$$A(\theta) = R^2 \left(\sin(2\theta) + \frac{1}{2} \sin(4\theta) \right); \quad 0 \leq \theta \leq \frac{\pi}{4}$$

$$A'(\theta) = R^2 (2\cos(2\theta) + 2\cos(4\theta))$$

$$0 = \cos(2\theta) + \cos(4\theta)$$

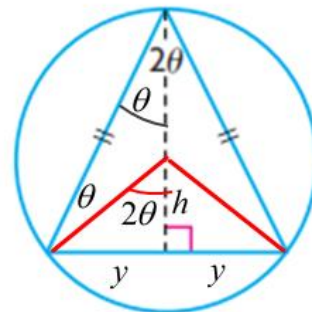
$$0 = \cos(2\theta) + 2\cos^2(2\theta) - 1$$

$$0 = (2\cos(2\theta) - 1)(\cos(2\theta) + 1)$$

$$\cos(2\theta) = \frac{1}{2} \quad \text{or} \quad \cos(2\theta) = -1$$

$$2\theta = \frac{\pi}{3} \quad \text{or} \quad 2\theta = \pi$$

$$\boxed{\theta = \frac{\pi}{6}} \quad \text{or} \quad \theta = \frac{\pi}{2} \quad \text{not in the domain}$$



5-10 Warm Up

① Differentiate:

a) $y = e^{4x} \sin(2x)$ b) $y = \frac{\cos^2(x)}{e^{-3x^2}}$ c) $y = x \log(x)$ d) $y = \frac{1 + \log_3(x)}{x}$

② Find the absolute max and min value(s) of the function $f(x) = xe^{-2x}$ on the interval $[0, \ln 3]$.

③ Find the equation of the tangent to the curve defined by $y = x + \ln(x)$ that is perpendicular to the line $3x + 9y = 8$.

④ Solve: $x + \sin(x) = e^x \cdot \cos(x)$ in the interval $x \in [0, 2\pi]$.

Solutions:

① a) $y' = 4e^{4x} \cdot \sin(2x) + e^{4x} \cdot \cos(2x) \cdot 2$
 $= 2e^{4x} [2\sin(2x) + \cos(2x)]$

b) $y = \cos^2(x) \cdot e^{3x^2}$
 $y' = 2\cos(x) \cdot (-\sin x) e^{3x^2} + \cos^2(x) \cdot e^{3x^2} \cdot 6x$
 $= -2\cos(x) \cdot e^{3x^2} [\sin x - 3x \cos(x)]$

c) $y' = \log(x) + \frac{x}{x \ln(10)}$
 $= \log(x) + \frac{1}{\ln(10)}$

d) $y' = \frac{\frac{1}{x \ln(3)} \cdot x - [1 + \log_3(x)]}{x^2}$
 $= \frac{\frac{1}{\ln(3)} - 1 - \log_3(x)}{x^2}$
 $= \frac{1 - \ln(3) - \ln(3) \cdot \log_3(x)}{\ln(3) \cdot x^2}$

5-10 Warm Up solutions

$$\begin{aligned} \textcircled{2} \quad f'(x) &= e^{-2x} + x \cdot e^{-2x} \cdot (-2) \\ &= e^{-2x} - 2x e^{-2x} \\ &= e^{-2x} (1 - 2x) \end{aligned}$$

critical #s $\rightarrow f'(x) = 0$

$$0 = e^{-2x} (1 - 2x)$$

$$e^{-2x} = 0 \quad \text{or} \quad 1 - 2x = 0$$

$$x = \text{undefined} \quad x = \frac{1}{2}$$

$$\begin{aligned} f(0) &= (0) e^{-2(0)} \\ &= 0 \end{aligned}$$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right) e^{-2\left(\frac{1}{2}\right)} \\ &\doteq 0.184 \end{aligned}$$

$$\begin{aligned} f(\ln(3)) &= (\ln(3)) e^{-2\ln(3)} \\ &\doteq 0.122 \end{aligned}$$

$\therefore$ The absolute max is approx. 0.184
at $x = \frac{1}{2}$ and the absolute min is
0 at $x = 0$

$$\begin{aligned} \textcircled{3} \quad 3x + 9y &= 8 \\ 9y &= -3x + 8 \\ y &= -\frac{1}{3}x + \frac{8}{9} \end{aligned}$$

$$\therefore m_{\perp} = 3$$

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$

$$3 = 1 + \frac{1}{x}$$

$$2 = \frac{1}{x}$$

$$x = \frac{1}{2}$$

$$\text{At } x = \frac{1}{2},$$

$$y = \left(\frac{1}{2}\right) + \ln\left(\frac{1}{2}\right)$$

$$\therefore y - \ln\left(\frac{1}{2}\right) - \frac{1}{2} = 3\left(x - \frac{1}{2}\right)$$

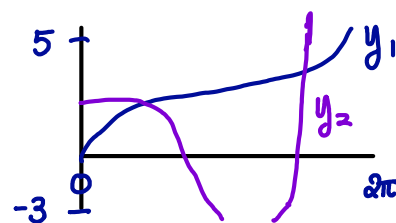
$$y = 3x - \frac{3}{2} + \frac{1}{2} + \ln\left(\frac{1}{2}\right)$$

$$y = 3x + \ln\left(\frac{1}{2}\right) - 1$$

$$\textcircled{4} \quad \text{graph } \begin{aligned} y_1 &= x + \sin(x) \\ y_2 &= e^x \cdot \cos(x) \end{aligned}$$

pt of intersection:

$$x_1 \doteq 0.819, \quad x_2 \doteq 4.74$$



Warm Up

1. Differentiate-Do not simplify the answer:

a) $y = e^{x^2} \log_2 \left[\sqrt[3]{4+x^2} \right]$

$$y' = 2xe^{x^2} \cdot \log_2 \left[\sqrt[3]{4+x^2} \right] + e^{x^2} \frac{\frac{1}{3}(4+x^2)^{-\frac{2}{3}} \cdot (2x)}{\sqrt[3]{4+x^2} \cdot \ln(2)}$$

b) $f(x) = \left[\frac{\sec(x^2)}{1+\cos(x)} \right]^4$

$$f'(x) = 4 \left[\frac{\sec(x^2)}{1+\cos(x)} \right]^3 \left[\frac{\sec(x^2)\tan(x^2) \cdot (2x) \cdot (1+\cos(x)) - \sec(x^2)[- \sin(x)]}{[1+\cos(x)]^2} \right]$$

2. Determine the slope of the normal line to the curve $\frac{x^2 + \sin(3y)}{x+y} = 3^{y^2}$ at point (1,0).

$$\ln \left[\frac{x^2 + \sin(3y)}{x+y} \right] = \ln(3^{y^2})$$

$$\text{At } (1,0): \frac{2(1) + \cos[3(0)] \cdot 3 \frac{dy}{dx}}{(1)^2 + \sin[3(0)]} - \frac{1 + \frac{dy}{dx}}{(1)+(0)} = 2(0) \frac{dy}{dx} \cdot \ln(3)$$

$$\ln[x^2 + \sin(3y)] - \ln(x+y) = y^2(\ln 3)$$

$$\frac{2x + \cos(3y) \cdot 3 \frac{dy}{dx}}{x^2 + \sin(3y)} - \frac{1 + \frac{dy}{dx}}{x+y} = 2y \frac{dy}{dx} \cdot (\ln 3)$$

$$\frac{2 + 3 \frac{dy}{dx}}{1+0} = 0 \rightarrow \frac{dy}{dx} = -\frac{2}{3} \therefore m_n = \frac{3}{2}$$

3. Using logarithmic differentiation, determine the derivative of $y = [\tan(4+x^2)]^{x+3}$.

$$\ln(y) = (x+3) \ln[\tan(4+x^2)]$$

$$\frac{1}{y} \frac{dy}{dx} = \ln[\tan(4+x^2)] + \frac{(x+3) \cdot \sec^2(4+x^2) \cdot (2x)}{\tan(4+x^2)}$$

$$\frac{dy}{dx} = [\tan(4+x^2)]^{x+3} \left[\ln[\tan(4+x^2)] + \frac{2x(x+3)\sec^2(4+x^2)}{\tan(4+x^2)} \right]$$

Equations of Tangents and Normals

Recall: The normal to the graph of $y = f(x)$ at point P is the line that is **perpendicular** to the tangent at P. The slope of the normal is the **negative reciprocal** of the slope of the tangent.

Example 1: Determine the equation of the normal to the curve $y = 2xe^x$ at the point (0, 0).

$$f(x) = 2xe^x$$

$$f'(x) = 2e^x + 2xe^x$$

$$f'(0) = 2$$

$$m_{\perp} = -\frac{1}{2}$$

$\therefore$ The equation of the normal is :

$$y - 0 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x$$

Example 2: Find the equation of the tangent to the curve $f(x) = \ln(e^x + e^{2x})$ at the point where $x = 0$.

$$f'(x) = \frac{e^x + 2e^{2x}}{e^x + e^{2x}}$$

$$= \frac{e^x(1 + 2e^x)}{e^x(1 + e^x)}$$

$$= \frac{1 + 2e^x}{1 + e^x}$$

$$f'(0) = \frac{1 + 2e^0}{1 + e^0}$$

$$= \frac{3}{2}$$

$$f(0) = \ln(e^0 + e^0)$$
$$= \ln 2$$

$\therefore$ The equation of the tangent is :

$$y - \ln 2 = \frac{3}{2}(x - 0)$$

$$y = \frac{3}{2}x + \ln 2$$

Example 3: A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle is given by $C(x) = 4 \ln x + \left(\frac{30-x}{10}\right)^2$ with a minimum production capacity of 10 kettles per day. How many kettles should be manufactured to keep the cost per kettle to a minimum?

$$C(x) = 4 \ln x + \left(\frac{30-x}{10}\right)^2$$

$$C'(x) = \frac{4}{x} + 2\left(\frac{30-x}{10}\right)\left(-\frac{1}{10}\right)$$

$$= \frac{4}{x} - \left(\frac{30-x}{50}\right)$$

$$= \frac{200 - 30x + x^2}{50x}$$

$$\text{critical \#s} \Rightarrow C'(x) = 0$$

$$x^2 - 30x + 200 = 0$$

$$(x-20)(x-10) = 0$$

$$x = 20 \text{ or } x = 10$$

$$C(20) = 4 \ln 20 + \left(\frac{30-20}{10}\right)^2$$

$$\approx 12.98 \text{ min value}$$

$$C(10) = 4 \ln 10 + \left(\frac{30-10}{10}\right)^2$$

$$\approx 13.2$$

$\therefore$ 20 kettles should be manufactured.

Example 4: Find the equation of the tangent line to the graph $f(x) = \sin^2 x$ at the point where $x = \frac{\pi}{3}$.

$$f(x) = \sin^2 x$$

$$f'(x) = 2 \sin x \cos x$$

$$f'\left(\frac{\pi}{3}\right) = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{3}$$

$$= 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{3}}{2}$$

$$y = \sin^2\left(\frac{\pi}{3}\right)$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{3}{4}$$

$\therefore$ The equation of the tangent is:

$$y - \frac{3}{4} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$y = \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\pi}{6} + \frac{3}{4}$$

$$12y = 6\sqrt{3}x - 2\sqrt{3}\pi + 9$$

$$6\sqrt{3}x - 12y - 2\sqrt{3}\pi + 9 = 0$$

Practice: Sinusoidal, Exponential and Logarithmic Functions

1. Differentiate and simplify fully.

a) $f(t) = \sqrt{2 + \sin^2 5t}$

$$\begin{aligned} f'(t) &= \frac{1}{2}(2 + \sin^2 5t)^{-\frac{1}{2}}(2 \sin 5t)(5) \\ &= \frac{5 \sin 5t \cos 5t}{\sqrt{2 + \sin^2 5t}} \end{aligned}$$

c) $f(t) = 3e^{\sin 2t}$

$$\begin{aligned} f'(t) &= 3e^{\sin 2t} (2 \cos 2t) \\ &= 6(\cos 2t)e^{\sin 2t} \end{aligned}$$

e) $y = 2^x \log_2(x^4)$

$$\begin{aligned} Y' &= 2^x \ln 2 (\log_2 x^4) + \frac{2^x (4x^3)}{x^4 \ln 2} \\ &= 2^x \ln 2 \log_2 x^4 + \frac{4(2^x)}{x \ln 2} \\ &= \frac{2^x (x)(\ln 2) \log_2 x^4 + 4(2^x)}{x \ln 2} \\ &= \frac{2^x [x(\ln 2) \log_2 x^4 + 4]}{x \ln 2} \end{aligned}$$

b) $h(x) = \frac{x^2}{2 - \cos \pi x}$

$$\begin{aligned} h'(x) &= \frac{2x(2 - \cos \pi x) - x^2(\pi \sin \pi x)}{(2 - \cos \pi x)^2} \\ &= \frac{4x - 2x \cos \pi x - x^2 \pi \sin \pi x}{(2 - \cos \pi x)^2} \\ &= \frac{x(4 - 2 \cos \pi x - \pi x \sin \pi x)}{(2 - \cos \pi x)^2} \end{aligned}$$

d) $g(x) = e^{\sqrt{x}} \ln \sqrt{x}$

$$\begin{aligned} g'(x) &= e^{\sqrt{x}} \cdot \left(\frac{1}{2}x^{-\frac{1}{2}}\right) \cdot \ln \sqrt{x} + \frac{1}{\sqrt{x}} \cdot \frac{1}{2}x^{-\frac{1}{2}} \cdot e^{\sqrt{x}} \\ &= \frac{1}{2}x^{-1} e^{\sqrt{x}} (x^{\frac{1}{2}} \ln \sqrt{x} + 1) \\ &= \frac{e^{\sqrt{x}} (\sqrt{x} \ln \sqrt{x} + 1)}{2x} \end{aligned}$$

f) $h(u) = \frac{e^{3u}}{1 + e^{3u}}$

$$\begin{aligned} h'(u) &= \frac{3e^{3u}(1 + e^{3u}) - 3e^{3u}(e^{3u})}{(1 + e^{3u})^2} \\ &= \frac{3e^{3u}(1 + e^{3u} - e^{3u})}{(1 + e^{3u})^2} \\ &= \frac{3e^{3u}}{(1 + e^{3u})^2} \end{aligned}$$

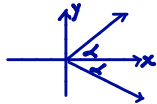
2. Find the maximum and minimum value of $f(x) = 2\sin x - x$, $0 \leq x \leq 2\pi$.

$$f'(x) = 2\cos x - 1$$

$$\text{critical \#s} \Rightarrow f'(x) = 0$$

$$2\cos x - 1 = 0$$

$$\cos x = \frac{1}{2}$$



$$x = \frac{\pi}{3}$$

$$\therefore x = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$

$$f(0) = 2\sin 0 - 0 = 0$$

$$f\left(\frac{\pi}{3}\right) = 2\sin \frac{\pi}{3} - \frac{\pi}{3}$$

$$= \sqrt{3} - \frac{\pi}{3}$$

$$\approx 0.685 \text{ max. value}$$

$$f\left(\frac{5\pi}{3}\right) = 2\sin \frac{5\pi}{3} - \frac{5\pi}{3}$$

$$= -\sqrt{3} - \frac{5\pi}{3}$$

$$\approx -6.97 \therefore \text{min value}$$

$$f(2\pi) = 2\sin 2\pi - 2\pi$$

$$= -2\pi$$

$$\approx -6.28$$

$\therefore$ The minimum value is $-\frac{3\sqrt{3}-5\pi}{3}$ and the maximum value is $\frac{3\sqrt{3}-\pi}{3}$.

3. Find the maximum and minimum value of $f(x) = xe^{-2x}$ on the interval $0 \leq x \leq \ln 3$.

$$f'(x) = e^{-2x} + xe^{-2x}(-2)$$

$$= e^{-2x}(1-2x)$$

$$= \frac{1-2x}{e^{2x}}$$

$$\text{critical \#} \Rightarrow f'(x) = 0$$

$$1-2x = 0$$

$$x = \frac{1}{2}$$

$$f'(x) \text{ DNE when } e^{2x} = 0$$

No solution

$$f(0) = 0 \text{ min value}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2}e^{-1} = \frac{1}{2e} \approx 0.184 \text{ max value}$$

$$f(\ln 3) = (\ln 3)e^{-2\ln 3}$$

$$= \frac{\ln 3}{e^{2\ln 3}} \approx 0.122$$

$\therefore$ The maximum value is $\frac{1}{2e}$ and the minimum value is 0.

4. Determine the point(s) of inflection for $f(w) = \frac{\ln w^2}{w}$.

$$f(w) = \frac{\ln w^2}{w}, w \neq 0$$

$$f'(w) = \frac{\frac{1}{w^2} \cdot 2w \cdot w - \ln w^2}{w^2}$$

$$= \frac{2 - \ln w^2}{w^2}$$

$$f''(w) = \frac{-\frac{1}{w^2} \cdot 2w \cdot w^2 - 2w(2 - \ln w^2)}{(w^2)^2}$$

$$= \frac{-2w - 2w(2 - \ln w^2)}{w^4}$$

$$= \frac{-2w(1+2 - \ln w^2)}{w^4}$$

$$= \frac{-2(3 - \ln w^2)}{w^3}$$

$$\text{Possible POI} \Rightarrow f''(w) = 0$$

$$3 - \ln w^2 = 0 \Rightarrow w^2 = e^3$$

$$3 = \ln w^2 \Rightarrow w = \pm \sqrt{e^3}$$

$$w = \pm 4.48$$

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	$(-\infty, -4.48)$	$(-4.48, 0)$	$(0, 4.48)$	$(4.48, \infty)$
-2	-	-	-	-
$3 - \ln w^2$	-	+	+	-
w^3	-	-	+	+
$f''(w)$	-	+	-	+
$f(w)$	CD	CU	CD	CU
	POI	w=0	POI	

$$f\left(e^{\frac{3}{2}}\right) = \frac{\ln\left(e^{\frac{3}{2}}\right)^2}{e^{\frac{3}{2}}}$$

$$= \frac{\ln e^3}{e^{\frac{3}{2}}}$$

$$= \frac{3}{e^{\frac{3}{2}}}$$

$$f\left(-e^{\frac{3}{2}}\right) = \frac{\ln\left(-e^{\frac{3}{2}}\right)^2}{-e^{\frac{3}{2}}}$$

$$= -\frac{3}{e^{\frac{3}{2}}}$$

$\therefore$ The points of inflection are $\left(e^{\frac{3}{2}}, \frac{3}{e^{\frac{3}{2}}}\right)$ and $\left(-e^{\frac{3}{2}}, -\frac{3}{e^{\frac{3}{2}}}\right)$

1. State in simplified form the derivative of the following:

a) $y = \frac{\cos 2x}{\sin 2x(\sin^2 x - \cos^2 x)}$

b) $y = (e^{-3\ln x})^2 + \log_2 2^{\cos x^2}$

c) $f(x) = (3x^2 + e)e^{-x}$

d) $y = \frac{\sin(3x^2 + 1)}{\cos(3x^2 - 1)}$

2. Determine the local extrema of the function $f(x) = \ln \sqrt{x^3 - x}$.

3. The position of a particle is given by $s(t) = 5 \sin\left(2t - \frac{\pi}{3}\right)$, $0 \leq t \leq 2\pi$, where t is in seconds and s is meters. Determine when the particle is at rest.

4. Find the point(s) on the graph of function $f(x) = (\ln x)^2 (\ln x - 1)$ that the tangent line is horizontal.

5. Find the values of a and b such that $f(x)$ is differentiable everywhere

$$f(x) = \begin{cases} \frac{\sin x}{\sin x - \cos x} & , \quad x \leq \frac{\pi}{2} \\ ax + b & , \quad x > \frac{\pi}{2} \end{cases}$$

6. Given $y = Axe^x + Be^x$, where A and B are constants, show that $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} = -y$.

7. Determine the point(s) of inflection of the function $f(x) = \frac{1}{4}\cos(2x) + \sin(x)$ over the interval $x \in [0, 2\pi]$.

8. Determine the equation of the normal to the function $f(x) = 2x - (\ln x)^2$ at the point with $x=e$.

9. If $f(x) = e^{\cos(2x)}$ on $[-\pi, \pi]$ determine and classify all local extrema.

10. For what value(s) of k does $y = e^{kx} \sin(x)$ satisfy $\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$?

Unit 5: Review-Solutions

Page 1

$$1. a) y = \frac{\cos(2x)}{\sin(2x) \cdot (-\cos(2x))}$$

$$= \frac{-1}{\sin(2x)}$$

$$= -\csc(2x)$$

$$y' = 2\csc(2x) \cot(2x)$$

$$b) y = (e^{-3\ln x})^2 + \log_2 2^{\cos x^2} \leftarrow y = e^{2\ln(x^{-6})} + \log_2 2^{\cos(x^2)}$$

$$y = x^{-6} + \cos(x^2)$$

$$y' = -6x^{-7} - 2x \sin(x^2)$$

$$y' = \frac{-6}{x^7} - 2x \sin(x^2)$$

$$c) f(x) = (3x^2 + e)e^{-x}$$

$$f'(x) = 6xe^{-x} - e^{-x}(3x^2 + e)$$

$$= e^{-x}(6x - 3x^2 - e)$$

$$= -e^{-x}(3x^2 - 6x + e)$$

$$d) y = \frac{\sin(3x^2 + 1)}{\cos(3x^2 - 1)}$$

$$y' = \frac{6x \cos(3x^2 + 1) \cos(3x^2 - 1) + 6x \sin(3x^2 - 1) \sin(3x^2 + 1)}{\cos^2(3x^2 - 1)}$$
$$= 6x \left[\frac{\cos(3x^2 + 1) \cos(3x^2 - 1) + \sin(3x^2 - 1) \sin(3x^2 + 1)}{\cos^2(3x^2 - 1)} \right]$$

Page 2 2. $f(x) = \ln \sqrt{x^3 - x}$

$$f(x) = \frac{1}{2} \ln(x^3 - x)$$

$$x^3 - x > 0$$

$$x(x-1)(x+1) > 0$$

$$\leftarrow \begin{array}{c} - \quad + \quad - \quad + \\ -1 \quad 0 \quad 1 \end{array} \rightarrow$$

$$D_f = (-1, 0) \cup (1, \infty)$$

$$f'(x) = \frac{1}{2} \left(\frac{3x^2 - 1}{x^3 - x} \right)$$

$$3x^2 - 1 = 0$$

$$x^2 = \frac{1}{3}$$

$$x = \pm \frac{\sqrt{3}}{3}$$

$$x = \frac{\sqrt{3}}{3} \text{ not in domain}$$

$$\begin{array}{c} + \quad - \quad + \\ -1 \quad -\frac{\sqrt{3}}{3} \quad 0 \quad \frac{\sqrt{3}}{3} \quad 1 \quad \infty \end{array}$$

$$\text{local max} \left(-\frac{\sqrt{3}}{3}, -0.477 \right)$$

3. $S(t) = 5 \sin(2t - \frac{\pi}{3}), 0 \leq t \leq 2\pi$

$$V(t) = S'(t) = 10 \cos(2t - \frac{\pi}{3})$$

$$\cos(2t - \frac{\pi}{3}) = 0$$

$$2t - \frac{\pi}{3} = k\pi + \frac{\pi}{2}$$

$$2t = k\pi + \frac{5\pi}{6}$$

$$t = \frac{k\pi}{2} + \frac{5\pi}{12}$$

→ OR specific solutions:

$$2t_1 - \frac{\pi}{3} = \frac{\pi}{2} \quad \text{or} \quad 2t_2 - \frac{\pi}{3} = \frac{3\pi}{2}$$

$$2t_1 = \frac{5\pi}{6}$$

$$t_1 = \frac{5\pi}{12}$$

$$t_2 = \frac{11\pi}{12}$$

$$t_3 = t_1 + \pi \leftarrow \text{period}$$

$$= \frac{5\pi}{12} + \pi$$

$$= \frac{17\pi}{12}$$

$$t_4 = \frac{11\pi}{12} + \pi$$

$$= \frac{23\pi}{12}$$

$$t = \frac{(6K+5)\pi}{12}$$

$$K = 0, 1, 2, 3$$

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$$\left(\frac{6K+5}{12}\right)\pi \leq 2\pi$$

$$6K+5 \leq 24$$

$$6K \leq 19$$

$$K \leq 3$$

K	t
0	$\frac{5\pi}{12}$
1	$\frac{11\pi}{12}$
2	$\frac{17\pi}{12}$
3	$\frac{23\pi}{12}$

4. $f(x) = (\ln x)^2 (\ln x - 1)$ $D_f : (0, \infty)$

$$f'(x) = 2(\ln x) \left(\frac{1}{x}\right) (\ln x - 1) + \frac{1}{x} (\ln x)^2$$

$$= \frac{1}{x} (\ln x) (2 \ln x - 2 + \ln x)$$

$$= \frac{\ln x}{x} (3 \ln x - 2)$$

$$f'(x) = 0 : \ln x = 0$$

$$x = 1$$

$$3 \ln x - 2 = 0$$

$$\ln x = \frac{2}{3}$$

$$x = e^{\frac{2}{3}}$$

∴ points are $(1, 0)$, $(e^{\frac{2}{3}},)$

page 4

$$5. \quad f'(\frac{\pi}{2}^-) = f'(\frac{\pi}{2}^+)$$

$$f'(x) = \begin{cases} \frac{\cos x (\sin x - \cos x) - \sin x (\cos x + \sin x)}{(\sin x - \cos x)^2} & x \leq \frac{\pi}{2} \\ a & x > \frac{\pi}{2} \end{cases}$$

$$\therefore f'(\frac{\pi}{2}) = -1$$

$$\therefore a = -1$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} f(x) = \lim_{x \rightarrow \frac{\pi}{2}^+} f(x) = f(\frac{\pi}{2})$$

$$1 = a(\frac{\pi}{2}) + b$$

$$1 = -\frac{\pi}{2} + b$$

$$b = 1 + \frac{\pi}{2}$$

$$6. \quad y = Ax e^x + Be^x$$

$$y'' - 2y' = -y$$

$$y' = A(e^x + x e^x) + B e^x$$

$$y' = A e^x (1+x) + B e^x$$

$$y'' = A[e^x(1+x) + e^x] + B e^x$$

$$= A e^x (2+x) + B e^x$$

$$\text{L.S.: } y'' - 2y' = 2A e^x + A x e^x + B e^x$$

$$- 2(A e^x + A x e^x + B e^x)$$

$$= -A x e^x - B e^x$$

page 5 7. $f(x) = \frac{1}{4} \cos(2x) + \sin(x) \quad x \in [0, 2\pi]$

$$f'(x) = -\frac{1}{2} \sin(2x) + \cos(x)$$

$$f''(x) = -\cos(2x) - \sin(x)$$

$$\cos(2x) + \sin(x) = 0$$

$$1 - 2\sin^2(x) + \sin(x) = 0$$

$$2\sin^2(x) - \sin(x) - 1 = 0$$

$$(2\sin(x) + 1)(\sin(x) - 1) = 0$$

$$\sin(x) = -\frac{1}{2} \quad \text{or} \quad \sin(x) = 1$$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6} \quad \text{or} \quad x = \frac{\pi}{2}$$

∴ Points of inflection: $(\frac{7\pi}{6}, \quad), (\frac{11\pi}{6}, \quad), (\frac{\pi}{2}, \quad)$

8. $f(x) = 2x - (\ln x)^2$

$$x = e$$

$$f(e) = 2e - 1$$

$$f'(x) = 2 - 2\ln(x) \left(\frac{1}{x}\right)$$

$$m_{\pm} = f'(e) = 2 - \frac{2}{e}$$

$$= \frac{2e-2}{e}$$

$$m_{\perp} = \frac{-e}{2e-2}$$

$$y - (2e-1) = \frac{-e}{2e-2} (x-e)$$

$$y = \frac{-e}{2e-2} x + \frac{e^2}{2e-2} + 2e-1$$

Page 6. 9. $f(x) = e^{\cos(2x)} ; [-\pi, \pi]$

$$f'(x) = -2 \sin(2x) e^{\cos(2x)}$$

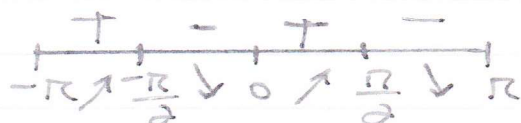
$$-2 \sin(2x) = 0 \quad \text{or} \quad e^{\cos(2x)} = 0$$

$\sin(2x) = 0$ no solution

$$2x = k\pi$$

$$x = \frac{k\pi}{2} \quad k = -2, -1, 0, 1, 2$$

$$x = -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi$$



$$\text{local max} \left(-\frac{\pi}{2}, \frac{1}{e} \right), \left(\frac{\pi}{2}, \frac{1}{e} \right)$$

$$\text{local min} (0, 1)$$

10. $y = e^{kx} \sin(x)$

$$y'' + 2y' + 2y = 0$$

$$y' = k e^{kx} \sin(x) + e^{kx} \cos(x)$$

$$= e^{kx} (k \sin(x) + \cos(x))$$

$$y'' = k e^{kx} (k \sin(x) + \cos(x))$$

$$+ e^{kx} (k \cos(x) - \sin(x))$$

$$= e^{kx} [k^2 \sin(x) + k \cos(x) + k \cos(x) - \sin(x)]$$

$$= e^{kx} [k^2 \sin(x) - \sin(x) + 2k \cos(x)]$$

$$y'' + 2y' + 2y = 0$$

$$e^{kx} [k^2 \sin(x) - \sin(x) + 2k \cos(x)] + 2k e^{kx} \sin(x) + 2e^{kx} \cos(x) + 2e^{kx} \sin(x) = 0$$

$$e^{kx} [k^2 \sin(x) - \sin(x) + 2k \cos(x) + 2k \sin(x) + 2 \cos(x) + 2 \sin(x)] = 0$$

$$e^{kx} [(k^2 + 1 + 2k) \sin(x) + (2k + 2) \cos(x)] = 0$$

solve for k where the coefficient is 0 to make the expression equal to 0.

$$k^2 + 2k + 1 = 0$$

$$(k+1)^2 = 0$$

$$k = -1$$

$$2k + 2 = 0$$

$$2k = -2$$

$$k = -1$$