Unit 3
Derivatives and Applications
Coursepack

Recall: What is a Derivative? The derivative of the function *f* at the number *a* is the slope of the curve $y = f(x)$ at $x = a$. The symbol for the derivative of *f* at the number *a* is $f'(a)$. Putting this together with the answers to our two questions above, we get

> *The derivative of f at the number a is given by*
>
> $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This leads to the definition of the derivative function

> *The derivative of f(x) with respect to x is the function* $f'(x)$*, where*
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Other Notation for Derivatives

Other notations for the derivative of the function $y = f(x)$ are $f'(x)$, y', and $\frac{dy}{dx}$

More About Derivatives

Since the derivative $f'(a)$ can be interpreted as the slope of the tangent at $(a, f(a))$, it follows that the derivative $f'(a)$ can also be considered the instantaneous rate of change of $f(x)$ with respect to *x* when $x = a$.

The Power Rule

> If $f(x) = x^n, n \in R$, then $f'(x) = nx^{n-1}$.
>
> Stated in Leibniz notation, if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$.

The Constant Multiple Rule

> If $f(x) = kg(x), k \in R$, then $f'(x) = kg'(x)$.
>
> Stated in Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$.

The Sum Rule[1]

> If $f(x) = p(x) + q(x)$, where $p(x)$ and $q(x)$ are both differentiable functions, then $f'(x) = p'(x) + q'(x)$.
>
> Stated in Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$.

[1] A corollary of the constant multiple rule and the sum rule is that if $f(x)=p(x) - q(x)$, then $f'(x)=p'(x)-q'(x)$

Examples

1. Determine the derivatives of each of the following

a) $f(x) = 4x^5$

b) $g(x) = 11x^{\frac{5}{2}}$

c) $h(x) = 4x^3 - 3\sqrt{x}$

d) $k(x) = (5x-3)^2$

e) $m(x) = \dfrac{4x^5 - \pi x^7}{5x^3}$

f) $n(x) = 7x^4 + \sqrt{x} - 3x^{\frac{3}{2}} - \dfrac{2}{x^4} - 99$

2. Determine the equation of the tangent to the graph of $y = x^3 + 2x^2 - 4x + 1$ at $x = 4$

3. A cubic polynomial function, $f(x) = ax^3 + bx^2 + cx + d$,is given such $f'(0) = 0, f'(1) = 5, \quad f'(2) = 16$,find $f'(3)$.

4. Determine the point(s) where the tangent to the curve $f(x) = x^3 - 6x^2 + 7$:

a) Has a slope of -9

b) Is horizontal

5. Find the values of x so that the tangent to $f\left(x\right)=\dfrac{3}{\sqrt[3]{x}}$ is parallel to the line x + 16y + 3 =0

6. Find the values for *a* and *b* so that *f(x)* is **differentiable** for all *x*.

$$f(x)=\begin{cases}-x^3+2x^2+4 & x\le 1\\ ax+b & x>1\end{cases}$$

Ex. 2. Given the parabola $y = 10x - x^2 - 16$. Find the equations of the two tangents that pass through the point P(4, 12). (Hint: Graph the function)

Challenging Questions

1. Determine the derivatives of each of the following

a) $f(x) = -\frac{3}{4}\sqrt[4]{x^5} - \frac{4}{3\sqrt{x^3}} + \pi^2 x^3 - \frac{7}{3}$

b) $g(x) = 4\sqrt[4]{x^3}\left(\pi\sqrt[5]{x} - 2^3\pi\right)$

c) $g(x) = \frac{3x^4 + 2\pi x^3 - 5\sqrt[3]{x}}{4x^2}$

2. Find the slope of the tangents to $f(x) = x^2 - x + 4$ such that they pass through an exterior point P(3,2).

3. Determine the value of a, given that the line ax – 4y + 21 = 0 is tangent to the graph of $y = \frac{a}{x^2}$ at x = -2.

4. The tangent to the cubic function $y = x^3 - 6x^2 + 8x$ at point A (3,-3) intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.

5. Find the equations of the tangent lines to the parabola $y = x^2 + x$ that pass through the point (2, -3). Sketch the curve and tangents.

WARM -UP: DERIVATIVE RULES

1. Differentiate the following. Express the answers with positive exponents.

a) $f(x)=x^{-4}-\sqrt{2x}-\dfrac{5x}{\sqrt{x}}+8^{2}$

b) $g(x)=x^{-2}\left(x^{-1}+1\right)^{2}$

2 The equation of the tangent to $y=ax^{3}+kx+1$ is $y=4x+k$ at $x=1$. Find the values of a and k. At what point does this tangent intersect the curve again?

<u>The Product Rule</u>

If $p(x)=f(x)g(x)$, then $p'(x)=f'(x)g(x)+f(x)g'(x)$.

Restated in Leibniz notation,

If u and v are functions of x, $\frac{d}{dx}(uv)=\frac{du}{dx}v+u\frac{dv}{dx}$

<u>Proof of the Product Rule</u>

Suppose $p(x) = f(x)g(x)$. Then

$$p'(x)=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{(x+h)-x}$$

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)+0-f(x)g(x)}{h}$$

Just like multiplying by 1 is a powerful tool in math, so is adding 0
In this case, $0=-f(x)g(x+h)+f(x)g(x+h)$
Who said mathematicians aren't creative? ☺

$$=\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

Next, we'll factor out $g(x+h)$ from the first two terms of the numerator, and we'll factor out $f(x)$ from the last two terms of the numerator

$$=\lim_{h\to 0}\left\{\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h)+f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$=\lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0}g(x+h)+\lim_{h\to 0}f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$=f'(x)g(x)+f(x)g'(x)$$

Examples

1. Differentiate $h(x)=(x^3-2x)(3x^4+2x+8)$ using the product rule

2. Find the value of $f'(-1)$ for the function $f(x) = (3x^4 - 12x^2 + 4x - 9)(6x^7 - 4x^4 + 18)$

Find an expression for $p'(x)$ if $p(x) = f(x)g(x)h(x)$

$$p(x) = f(x)g(x)h(x)$$
$$= [f(x)g(x)]h(x)$$
$$p'(x) =$$

$$\boxed{p'(x) = \qquad\qquad\qquad\qquad}$$

This is called the ***extended product rule*** for three functions

3. Differentiate the rational function $f(x) = x(2x+5)(x-1)$ by using the extended product rule.

4. If $g(x) = x^2 f(x)$, $f(2) = -2$ and $g'(2) = 8$,then determine $f'(2)$.

The Power of a Function Rule for Integer Exponents

If u is a function of x, and n is an integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1} g'(x)$

We will prove a more general statement of this (the Chain Rule) in section 2.5.

Examples

1. Determine $h'(x)$ where $h(x) = (4x^2 - 3x + 1)^7$. Then, evaluate $h'(1)$.

2. Find the derivative of $g(x) = \left(3x^2 - 5\right)^6 \left(2x^3 + 1\right)^4$.

3. Differentiate the following. Write the final answer with positive exponents.

$$h(x) = \frac{(2x-1)^2}{(3x+2)^3}$$

Quotient Rule

If $h(x) = \dfrac{f(x)}{g(x)}$, then

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, g(x) \neq 0$$

In Leibniz notation, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$

Proof:

Since $h(x)=\dfrac{f(x)}{g(x)}, g(x)\neq 0$, therefore

$h(x)g(x)=f(x)$ → multiply each side by $g(x)$

$h'(x)g(x)+h(x)g'(x)=f'(x)$ →differentiate each side

$h'(x)g(x)=f'(x)-h(x)g'(x)$

$$h'(x)=\frac{f'(x)-h(x)g'(x)}{g(x)}$$

$$=\frac{f'(x)-\dfrac{f(x)}{g(x)}g'(x)}{g(x)}$$

$$=\frac{f'(x)-\dfrac{f(x)}{g(x)}g'(x)}{g(x)}\times\frac{g(x)}{g(x)}$$

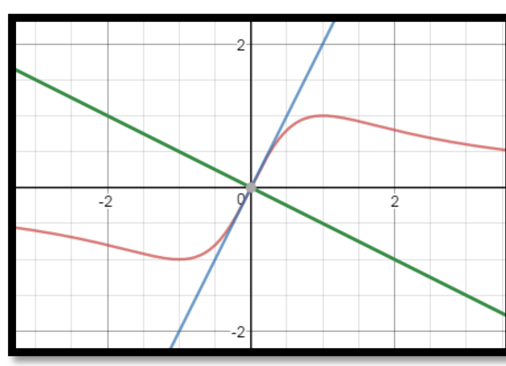
$$=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Examples

1. Determine the derivative of $h(x)=\dfrac{3x-4}{x^2+5}$

2. Determine the equation of the **normal** to $y=\dfrac{2x}{x^2+1}$ at $x=0$.

Definition: A normal line to the graph of a function f(x) is defined to be the line perpendicular to the tangent at a given point



3. Determine the coordinates of each point on the graph of $f(x)=\dfrac{2x+8}{\sqrt{x}}$ where the tangent is horizontal.

Challenging Questions

1. Determine the equation of the tangent line to $g(x)=\left(\dfrac{1}{x^3}+1\right)(x-1)$ at $x=-1$.

2. If $g(x)=\dfrac{f(x)}{\sqrt{x-1}}$, where $f(5)=8$, and $f'(5)=-5$, find $g'(5)$.

3. Find the points on the function $f(x)=\dfrac{x+9}{x+8}$ where the tangent lines pass through the origin

4. Recall: **A normal line to the graph of a function f(x) is defined to be the line perpendicular to the tangent at a given point.** Find the equation of the normal to the curve $y=\sqrt[3]{x^2-1}$ at the point where x=3 .

5. Let f and g be functions such that $g(x)=\dfrac{f(x)}{x}$. If $y=2x-3$ is the equation of the tangent to the graph of f(x) at x=1, what is the equation of the line tangent to the graph of g(x) at x=1 ?

6. Find the points on the curve $y=\dfrac{x}{x+1}$ where the **normal** line is parallel to $x+y=2$.

Warm-Up: PRODUCT RULE

The limit below represents the derivative of some function f(x) evaluated at some number ***a***. Determine the function and the number ***a***.

$$f'(a)=\lim_{h\to 0}\frac{2(6+h)^2-2(6)^2}{h}\quad,\quad f(x)=\_\_\_\_\_\_\_\_\_\_\quad,\quad \boldsymbol{a}=\_\_\_\_\_\_\_\_\_\_$$

1. Differentiate the following .Where applicable; write the final answers with positive exponents.

   a) $g(x)=\left(5\sqrt[5]{x^3}-\frac{1}{2x^2}\right)\sqrt[3]{x}$

   b) $g(t)=\frac{\pi t^5-2t^{-4}+3\pi^2}{3t^2}$

2. Given $f'(1)=4, g'(1)=-2, f(1)=1$, and $g(1)=1$, find $h'(1)$ if $h(x)=\left(2x-\sqrt{x}\right)^2 g(x)+x^3 f(x)$.

The Chain Rule

If $g(x)$ is differentiable at x and $f(x)$ is differentiable at $g(x)$, then the composite function, $h(x)=f(g(x))$ or $h(x)=(fog)(x)$ is differentiable at x and $h'(x)$ is given by the product

$$h'(x)=f'(g(x))g'(x).$$

In Leibniz notation, If y=f(u) and u=g(x), are both differentiable functions, then

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Proof of Chain Rule:

$$[f(g(x))]' = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}$$

$$=\lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{h}\times 1\right]$$

$$=\lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{h}\times\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right]$$

We can only make this move if we know that $g(x+h)-g(x)\neq 0$. In other words, this proof is not valid over any domain of the function for which the graph of $y=g(x)$ is a straight horizontal line.

$$=\lim_{h\to 0}\left[\left(\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right)\left(\frac{g(x+h)-g(x)}{h}\right)\right]$$

$$=\lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

Look at the denominator of the first fraction. We're taking the limit of that fraction as $h\to 0$. We know that $\lim_{h\to 0}[g(x+h)-g(x)]=0$. So, we'll let $g(x+h)-g(x)=k$. Recognizing that $k\to 0$ as $h\to 0$, we're able to rewrite that last line as follows:

$$=\lim_{k\to 0}\left[\frac{f(g(x)+k)-f(g(x))}{k}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$=f'(g(x))g'(x)$$

Examples:

1. If $y=u^2+u-1$, and if $u=x^2-2\sqrt{x}$, then evaluate $\frac{dy}{dx}$ at $x=1$

2. Determine the derivative of $f(x)=\sqrt[3]{\frac{x^2-3}{3-5x}}$

3. Differentiate:

a) $f(x)=m\left(nx^2+rx\right)^{\sqrt{7}}$

b) $f(x)=(5+3x)^{\pi}$

c) $y=\frac{x}{\sqrt{1-4x^2}}$

d) $\mathrm{f(x)}=\sqrt{4\mathrm{x}^2+6\mathrm{x}-1}$

e) $y=(2x^2-9)\sqrt{3x^2+5x}$

f) $m(t)=\sqrt[3]{t+\sqrt{1+t^2}}$

4. a) If $h(x)=\dfrac{\left(f(x)\right)^2}{g(x)}$, determine $h'(x)$.

b) Given $f(1)=2$, $f'(1)=-3$, $g(1)=1$ and $g'(1)=4$ find $h'(1)$.

5. If $y=f\left(3x^4\right)$ and $f'(3)=\dfrac{-1}{4}$, determine $\left.\dfrac{dy}{dx}\right|_{x=1}$.

Practice:

1. Given that $g(2)=4, g'(2)=-1, h(2)=2,$ and $h'(2)=3,$ find $f'(2)$ if

   a) $f(x)=(g(x))^3$ b) $f(x)=g(h(x))$

2. Let $f(x)=h(g(x))$ and $j(x)=h(x)g(x),$ where *g* and *h* are differentiable functions on $\mathbb{R}$. Fill in the missing entries on the table below and determine *h*(4) and *h′*(4).

x	h(x)	h′(x)	g(x)	g′(x)	f(x)	f′(x)	j(x)	j′(x)
0		**1**	**2**			**–8**	**–8**	**10**
1	**2**					**4**	**4**	**6**
2		**4**	**4**		**15**	**8**	**4**	**18**

3. Slope of the normal to the curve with equation $y = \boldsymbol{a}x + \dfrac{\boldsymbol{b}}{4-3x}$ at point $(1,6)$ is $-\dfrac{1}{2}$. Find the values of ***a*** and ***b***.

4. Find $\boldsymbol{k}$ given that the tangent to $f(x)=\dfrac{4}{(\boldsymbol{k}x+1)^2}$ at $x=0$ passes through (1, 0)

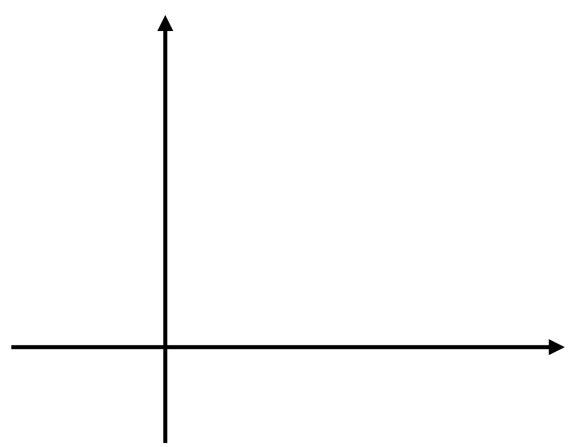
5. Consider $f(x)=\dfrac{4}{\sqrt{4-x}}$.

   (a) Find the equations of the tangent and normal at the point where P(3,4).

   (b) If the tangent line cuts the x-axis at A and the normal line cuts the x-axis at B, find the coordinates of A and B.

   (c) Find the area of triangle PAB.

.



More Practice on using Chain Rule

1. Differentiate the following .Where applicable; write the final answers with positive exponents.

a) $y=\left(2+x^2\right)^{\pi}+\sqrt[5]{1-x^2}+\dfrac{3x^2-\sqrt{x}+5}{\sqrt{5x}}$

b) $S(t)=t^2\left(1-\dfrac{2\pi}{t^2}\right)+\sqrt{4t^2-5}$

c) $h(x)=\dfrac{(3x+2)^{-3}}{(2x-1)^{-2}}$ ***(Express in a fully factored form)***

2. Let f and g be differentiable functions such that $g(1)=3, f(1)=-2$, $f(3)=-1$, $f'(3)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{f(g(x))}{f(x)+g(x)}$. If $h'(1)=5$, find the value of $f'(1)$.

3. Find the points on the curve $y=\left(1-\dfrac{x}{5}\right)^3$ where the slope of normal line is 15.

4. If $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=-4$ and $g(x)=f\left(\sqrt{5-x^2}\right)$, determine the value of $g'(1)$.

5 . Assume that $h(x)=\left[f(x)\right]^3.g(x)$, where f and g are differentiable functions

If $f(0)=\dfrac{-1}{2}, f'(0)=\dfrac{-8}{3}$ and $g(0)=-1, g'(0)=-2$, determine an equation of the line tangent to the graph of h at x=0 .

6. Consider the curve $\mathbf{y=a\sqrt{x}+\dfrac{b}{\sqrt{x}}}$ where ***a*** and ***b*** are constants. The normal to this curve at the point where x = 4 is 4x + *y* = 22. Find the values of *a* and *b*.

7. Line $y=k$ is tangent to the curve $f(x)=\dfrac{x^2-5}{x-(k+1)}$ at x=1. Find value of k.

8. Given $y=\dfrac{u+3}{2u-1}$, and $u=\sqrt{x^2+3}$, determine $\left.\dfrac{dy}{dx}\right|_{x=1}$ by using the Leibniz notation.

1. Let f and g be differentiable functions such that $g(1)=3, f(1)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{(fg)(x)}{f(x)+g(x)}$. If $h'(1)=5$, find the value of $f'(1)$.

2. Find the points on the curve $y=\left(1-\dfrac{x}{5}\right)^3$ where the slope of normal line is 15.

3. If $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=-4$ and $g(x)=f\left(\sqrt{5-x^2}\right)$, determine the value of $g'(1)$.

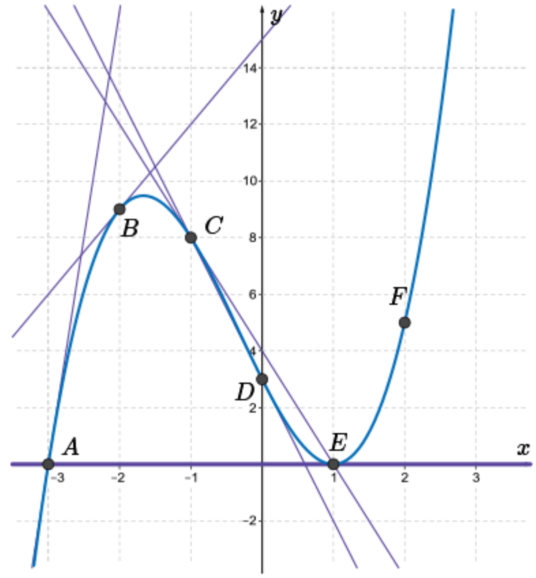
4. Assume that $h(x)=\left[f(x)\right]^3 . g(x)$, where f and g are differentiable functions.
   If $f(0)=\dfrac{-1}{2}, f'(0)=\dfrac{-8}{3}$ and $g(0)=-1, g'(0)=-2$, determine an equation of the line tangent to the graph of h at x=0 .

5. Let f and g be the functions satisfying $f(x)=(x)\sqrt{g(x)}$ for all real numbers x. If $y=4x-3$ is the equation of the tangent to the graph of g(x) at x=3, what is the equation of the line tangent to the graph of f(x) at x=3.

6. Determine the equation of the line **normal** to the curve $h(x)=\dfrac{(x+1)(2x-3)}{1-x}$ at $x=-1$.

7. Determine the point(s) (a, b) on the curve of $y=\dfrac{x+1}{x-2}$ where the slope of the tangent line is -3.

8. Line $y = k$ is tangent to the curve $f(x)=\dfrac{x^2-5}{x-(k+1)}$ at x =1. Find value of k.



So far, we have seen that the value of the derivative, $f'(x)$, gives us the instantaneous rate of change of a function, $f(x)$, at a point. It is represented graphically by the slope of the tangent line to the curve, $y = f(x)$, at that point. Throughout the graph, the slope of the tangent line is continually changing. We can describe this change as the rate of change of the slope of the tangent. To determine how the slope of the tangent is changing, we differentiate the derivative function f′(x). If f′(x) is differentiable, then the derivative of the derivative function can be found.

This is known as the **second derivative of** f(x), and is denoted in function notation as $f''(x)$.

In Leibniz notation, the second derivative is denoted as

$$\frac{d^2y}{dx^2} = \frac{d^2[f(x)]}{dx^2}.$$

Example 1

Find the second derivative of $f(x) = x^4 + 3x^2 - 5\sqrt{x}$

Example 2: Find $\frac{d^2y}{dx^2}$, given $y = \frac{5x - 3}{2x}$.

Third Derivatives

The second derivative of f(x) is found by taking the derivative of f(x) twice. This can be extended further if $f''(x)$ is differentiable; taking the derivative of $f''(x)$ gives the third derivative of f(x), which is denoted in function notation as $f'''(x)$ or $f^{(3)}(x)$.

In Leibniz notation, the third derivative is denoted as shown.

$$\frac{d^3y}{dx^3} = \frac{d^3\left[f(x)\right]}{dx^3}$$

Note that the brackets around the 3 are required in $f^{(3)}(x)$.

In general, if the derivatives remain differentiable, the n^{th} derivative of f(x) is found by taking its derivative n times, and is denoted $f^{(n)}(x)$.

Example 3: Determine the third derivative of the following functions

a) $y = \frac{1}{x}$

b) $y = \frac{3}{2x-6}$

c) $y = \sqrt{x}$

Example 4: Suppose $f(x)=ax^2+bx+c$ and $f(1)=8, f'(1)=3$, and $f''(1)=-4$. Determine a,b, and c.

APPLICATIONS OF HIGHER ORDER DERIVATIVES – LINEAR MOTION

Definitions

Position s(t) is the location of an object at a value of time t.

Velocity v(t) is the rate of change of position over time, so

$$v(t) = s'(t) = \frac{ds}{dt}$$

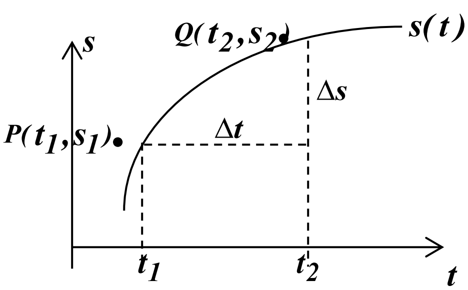
Acceleration a(t) is the rate of change of velocity over time, so

$$a(t) = v'(t) = s''(t)$$

Or

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Position



- $s(0)$ represents the initial position of the object (when $t = 0$)
- $s(t) > 0$ indicates that the object is to the right of the origin
- $s(t) < 0$ indicates that the object is to the left of the origin.
- $s(t) = 0$ indicates that the object is at the origin (where it started)

Velocity

Thus instantaneous velocity is $v(t) = s'(t) = \dfrac{ds}{dt}$

- $v(0)$ is the initial velocity (when $t = 0$)
- $v(t) > 0$ $\Rightarrow$ object is moving to the right (positive direction)
- $v(t) < 0$ $\Rightarrow$ object is moving to the left (negative direction)
- $v(t) = 0$ $\Rightarrow$ object is at rest or object may be changing directions or object may be at a maximum/minimum height

Acceleration

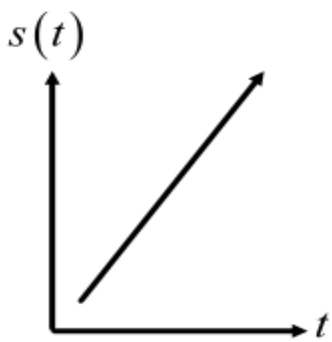
- $a(t) > 0$ $\Rightarrow$ object is accelerating (velocity is increasing)
- $a(t) < 0$ $\Rightarrow$ object is decelerating (velocity is decreasing)
- $a(t) = 0$ $\Rightarrow$ object is at a constant velocity or object is at a max/min velocity

If $sv > 0$ the object is moving away from the origin
If $sv < 0$ the object is moving towards the origin

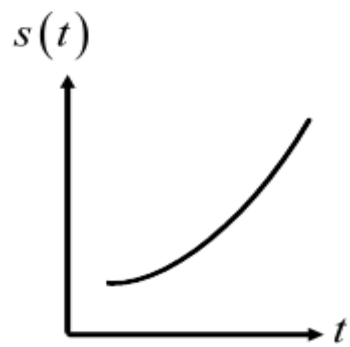
If $av > 0$ the object is speeding up
If $av < 0$ the object is slowing down

Example 5: In each graph determine if the object is moving away from/towards the origin or speeding up/slowing down

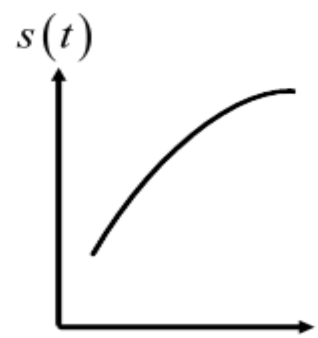
(a)



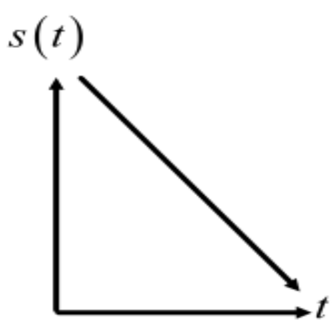
(b)



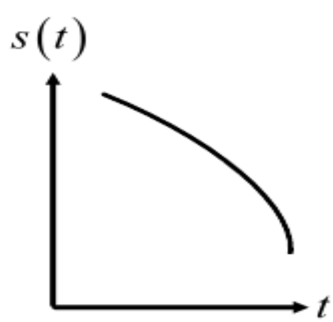
(c)



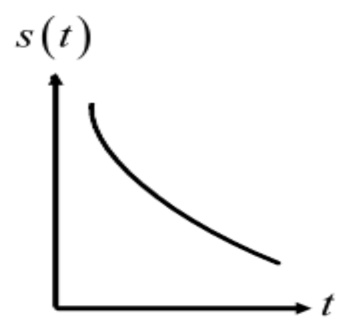
(d)



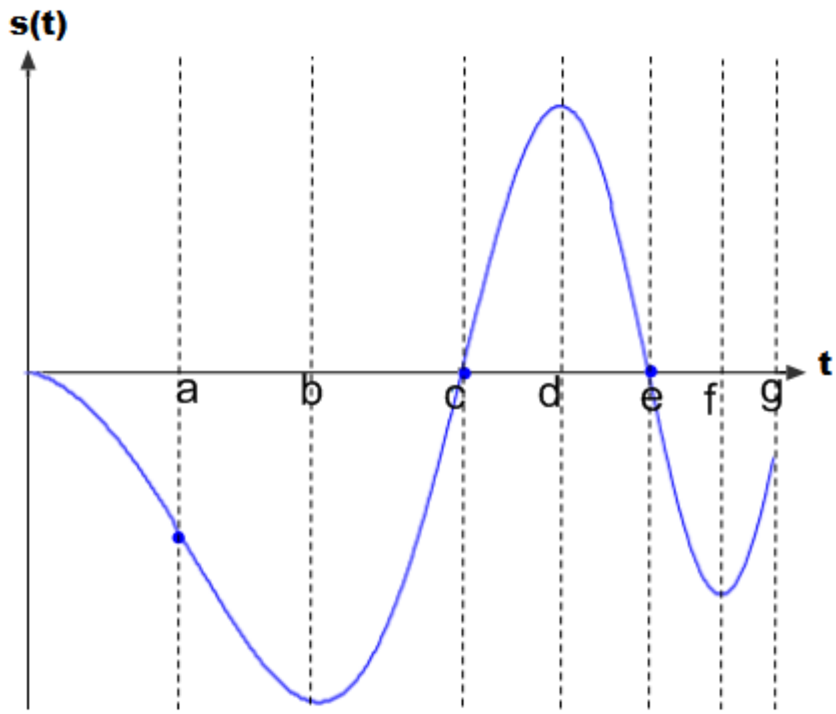
(e)



(f)



Example 6: Complete the table below the graph.



Interval	Displacement +/-	Velocity +/-	Acceleration +/-	Speeding up/ Slowing down	Moving away/ toward
[0,a]					
[a,b]					
[b,c]					
[c,d]					
[d,e]					
[e,f]					
[f,g]					

Example 7: An object moves in a straight line according to the motion equation

$$s(t) = 2t^3 - 15t^2 + 36t - 22 \, , \, t \geq 0 \, .$$

(a) Find the velocity and acceleration at time ***t***.

(b) Find the initial conditions and interpret their meaning.

(c) Find the average velocity and acceleration from ***t*** = 2 to ***t*** = 4.

(d) When is the particle at rest?

(e) When does the object move in a positive direction?

Example 8: The motion of a particle on straight line is given by position function $s(t)=36-24t+9t^2-t^3$, where ***s*** is in meter and ***t*** is in minute.

(a) Find the velocity and acceleration at time ***t***.

(b) After how many minutes does the object stop?

(c) When is the particle moving toward the motion detector?

(d) When is the object slowing down?

(e) Determine the total distance traveled in the first 7 minutes.

Application of Higher Order Derivatives: Linear Motion

	Displacement	Velocity	Acceleration
Notation:			
Unit of measure:			
Order of Derivative	n/a		
Definition:			
Other notes:			

1. Vertical Motion:

A ball is thrown up and its motion is described by $h(t) = -4.9t^2 + 6t + 2$ where h is the height in metres and t is the time in seconds.

(a) Find the velocity and acceleration function:

(d) Sketch $h(t), v(t)$ and $a(t)$ below each other.

(b) Find the initial velocity, v_0.

(c) When does the ball reach its max height?
What is this height?

(e) When does the ball hit the ground?

(f) What is the velocity (rate of change of height) upon impact?

(g) What is the acceleration (rate of change of velocity) upon impact?

(h) Determine the time interval when the ball is speeding up (accelerating) and when it is slowing down (decelerating)

Exit Card!

Given position function $s(t) = t^{\frac{5}{2}}(7-t),\ t \geq 0$.

(a) Is the particle speeding up or slowing down when $t = 4s$.

(b) At what time(s) is the object at rest?

(c) In which direction is the object moving at ***t =4***

(d) When is the object moving in a negative direction?

(e) When does the object return to its initial position?

Practice: Higher Order Derivatives

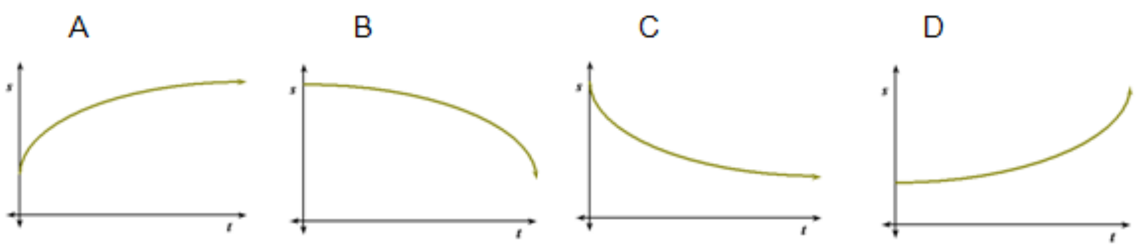
1. Find the first and second derivatives of each function.

a) $f(x) = 2x^4 - 4x^{-2}$ **b)** $y = \dfrac{3}{x^2}$

c) $y = \dfrac{2x+1}{x}$ **d)** $g(x) = (x-1)(x+1)^3$

g) $y = \dfrac{x^2-4}{x+1}$ **h)** $s(t) = t^3 + \dfrac{2}{\sqrt{t}}$

2. A boat demonstrates a positive velocity but a negative acceleration. Which of the following plots illustrates its position?



3. A particle moves on the y axis with this relationship between position and time: $s(t) = t^3 - 17t^2 + 80t - 100$. Determine the time interval(s) during which it is :

a) located below the origin

b) moving upward (moving in positive direction)

c) slowing down

d) moving away from the origin

4. A particle moves on the y axis with this relationship between position and time:

$$s(t) = \frac{1}{4}t^4 - 2t^3 + \frac{9}{2}t^2 - 4t + 2$$

a) Describe the motion of the particle at $t = 0$.

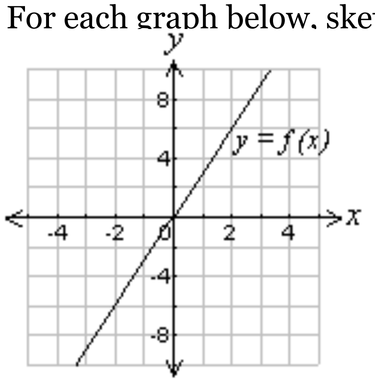
b) What is the average velocity of the particle between $t = 1$ and $t = 4$?

c) When does the particle reverse direction?

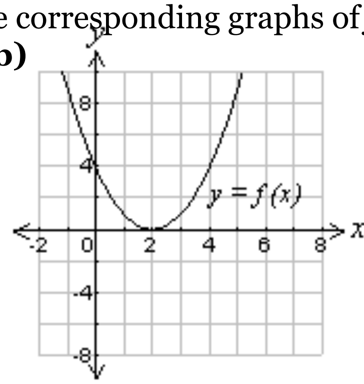
d) Find the total distance traveled from $t = 0$ to $t = 5$.

5. For each graph below, sketch the corresponding graphs of f' and f''.

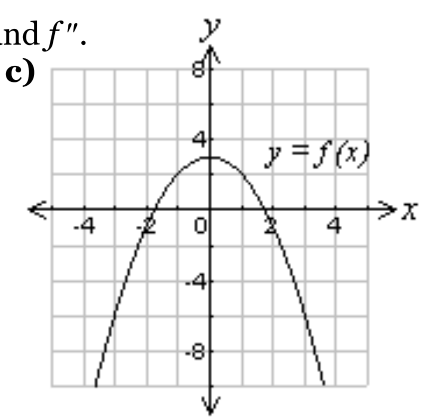
a)



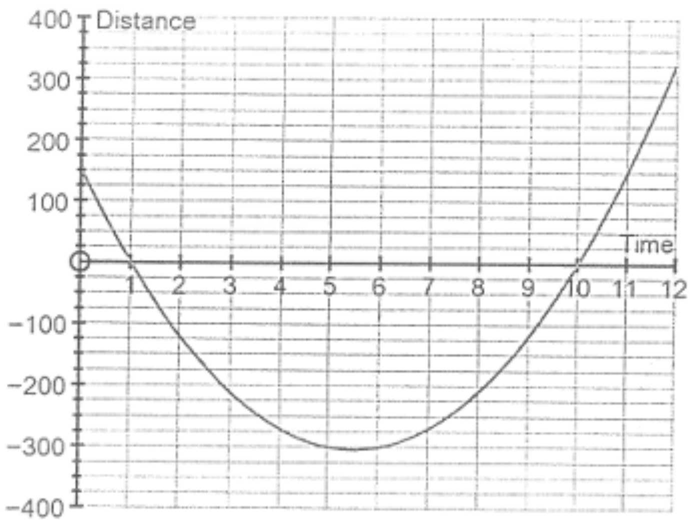
b)



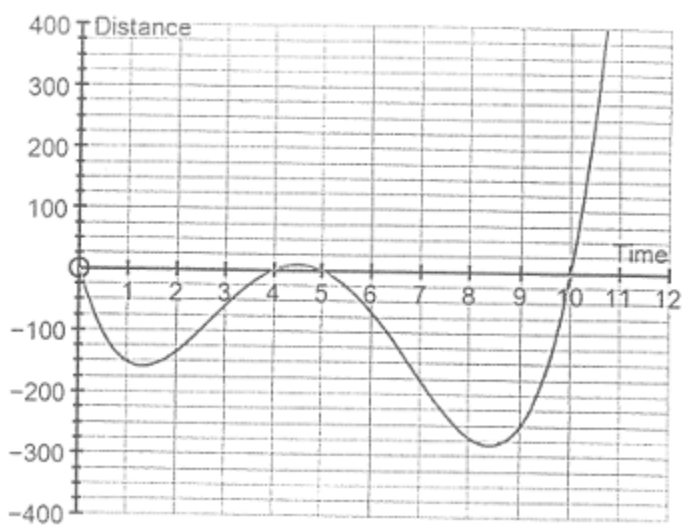
c)



d)



e)



6. For what values of the constants a, b, c, and d does the function $f(x) = ax^3 + bx^2 + cx + d$ satisfy both of the following conditions?

i) $f''(0) = 0$ at the origin **ii)** a horizontal tangent at $(2, 4)$

7. A person's height, in metres, can be modelled by the function $h(t) = \dfrac{at}{b+t} + c$, where t is the age of the person, in years, and a, b, and c are positive constants.

a) $h'(t)$ represents the growth rate. What does $h''(t)$ represent?

b) Show that $h''(t)$ is always negative. What does this indicate about the growth rate?

c) Show that

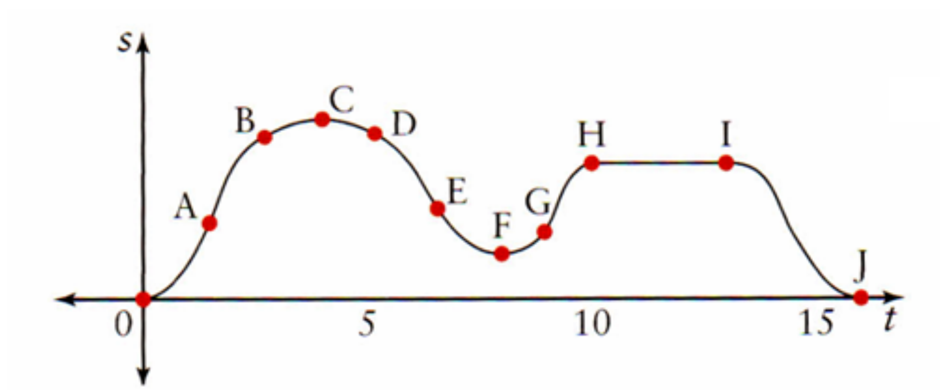
i) the initial height is c

ii) the initial growth rate is $\dfrac{a}{b}$

d) Suggest reasonable values for the constants.

e) In what way(s) is the function not a realistic model for the height of a person?

8. The following graph shows the position function of a bus during 15 min trip.



a) What is the initial velocity of the bus?

b) What is the bus's velocity at C and F?

c) Is the bus going faster at A or at B? Explain.

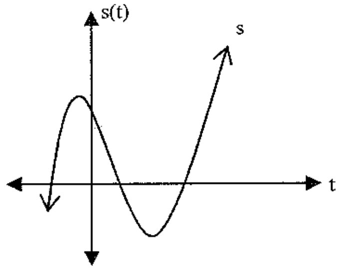
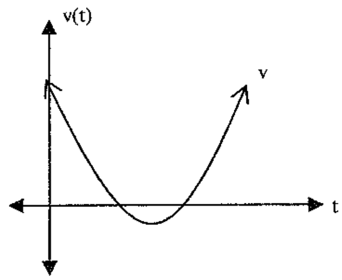
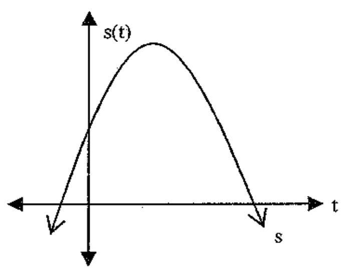
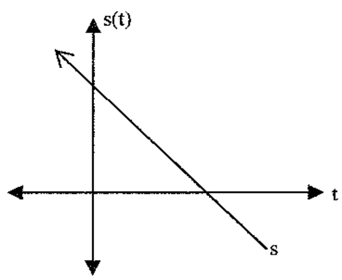
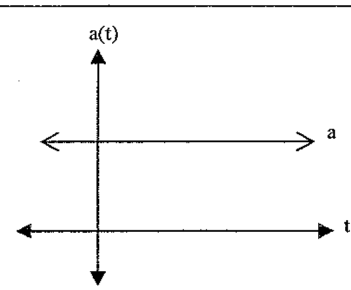
d) What happens to the motion of the bus between H and I?

e) What happens at J?

9. Refer to the graph in question 8. Is the acceleration positive, zero, or negative during the following intervals?

a) 0 to A **b)** C to D **c)** E to F **d)** G to H **e)** F to G

10. Create sketches so that each graph in a set corresponds to the other two.

	Displacement	**Velocity**	**Acceleration**
a)			
b)			
c)			
d)			
e)			

Warm Up

The forward motion of a space shuttle, t seconds after touchdown, is described by $s(t) = 189t - t^{\frac{7}{3}}$, where s is measured in metres.

a) How much time is needed for the shuttle to stop?

b) How far does the shuttle travel from touchdown to stop?

c) Does the shuttle speeding up or slowing down at 8 seconds after touchdown?

Implicit Differentiation

Until now, we have described functions by expressing one variable explicitly in terms of another variable: $y = f(x)$. Example: $y = x^2$, $y = \sqrt{1-x^2}$, $y = x^3 - 3x$.

However, curves may be defined by a relation such as:

$$x^2 + y^2 = 9$$

$$\frac{x^2}{4} + y^2 = 1$$

$$y^5 + x^2 y - 2x^2 = -1$$

Use a method called **implicit differentiation**.

Strategy: differentiate both sides of the equation with respect to x and then solve for $\frac{dy}{dx}$ or y'.

(Note: use chain rule when differentiating terms containing y.)

Recall: $\frac{dx^2}{dx} = 2x$ and $\frac{dh^2}{dh} = 2h$

However: $\frac{dy^2}{dx} = 2y\frac{dy}{dx}$

Ex. 1. Differentiate:

$$\frac{x^2}{4} + y^2 = 1$$

Ex. 2. Differentiate:

$$y^5 + x^2 y - 2x^2 = 1$$

Ex. 3. Find the equation of the tangent to the curve $3x^4 + x^2y^2 + y^3 = 5$ at the point (1, 1).

Ex. 4. If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$. Verify that any tangent line to this circle is perpendicular to the radius at the point of tangency.

Ex. 5. Find the slope of the tangent line to the curve $y = \sqrt[3]{6 + x^2}$ at the point $(\sqrt{2}, 2)$.

Ex. 6. Find the equation of the tangent to the curve $\sqrt{y} = \sqrt{xy} - 2\sqrt{x}$ at $x = 4$.

Implicit Differentiation

Explicit form	**Implicit form**
$y = \frac{2}{x}$	$xy = 2$

Example 1:

Find the derivative of both $y = \frac{2}{x}$ and $xy = 2$ using explicit and implicit differentiation, respectively, and show that the two have equivalent derivative functions $\frac{dy}{dx}$.

Most of the time, we use implicit differentiation when we're dealing with curves that are not functions, with powers of y greater than 1. This is why we cannot solve for y explicitly. We will, however, be able to solve for $\frac{dy}{dx}$ after differentiation implicitly.

Important note : When differentiation implicitly, you must show that you are taking the derivative of both sides with respect to x.

Example 2:

Find the slope of the graph of $y^3 + y^2 - 5y - x^2 = -4$ at $(1,-3)$ by (a) finding $\frac{dy}{dx}$ then evaluating it at $(1,-3)$, then by (b) differentiation and plugging in $(1,-3)$ before solving for $\frac{dy}{dx}$.

Example 3:

Find the coordinates where the graph $x^2 + y^2 = 25$ has horizontal and vertical tangent lines. At what point is the slope $\frac{3}{4}$?

Example 4:

Find the coordinates (x, y) of any horizontal and vertical tangent lines to the curve given by the equation $2x^2 + xy + 4y^2 = 3$. Be sure to label your work.

Example 5:

Determine the equation of the tangent line of $3\left(x^2 + y^2\right)^2 = 100xy$ at the point $(3,1)$

Practice-Implicit Differentiation

Show all work on a separate sheet of paper. No calculator unless otherwise stated.

1. Find $\frac{dy}{dx}$

   a) $x^3 - 3x^2y + 4xy^2 = 12$ b) $\sqrt{xy} = x + 3y$ c) $\left(y^2 + 2xy\right)^2 = 4(x+1)^2$

2. Find $\frac{dy}{dx}$ at the indicated point, then find the equation of both the tangent and normal lines.

   a) $y^2 = \frac{x^2 - 4}{x^2 + 4}$ at $(2,0)$ b) $(x+y)^3 = x^3 + y^3$ at $(-1,1)$

3. Find $\frac{d^2y}{dx^2}$ in terms of *x* and *y*.

   a) $x^2 + y^2 = 36$ b) $1 - xy = x - y$ c) $\sqrt[3]{x^2} + \sqrt[3]{y^2} = 1$

4. Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has either a horizontal or vertical tangent. Be sure to label which is which, if either exist.

5. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the *x*-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

6. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$

7. The slope of the tangent is -1 at the point $(0,1)$ on $x^3 - 6xy - ky^3 = a$, where *k* and *a* are constants. The values of the constants *a* and *k* are what?

Multiple Choice

_____ 8. Find y' when $xy+5x+2x^2=4$.

(A) $y'=\frac{5+2x-y}{x}$ (B) $y'=-\frac{y+5+4x}{x}$ (C) $y'=-(y+5+4x)$ (D) $y'=\frac{y+5+2x}{x}$

(E) $y'=-\frac{y+5+2x}{x}$ (F) $y'=\frac{y+5+4x}{x}$

_____ 9. Find $\frac{dy}{dx}$ when $\frac{3}{\sqrt{x}}+\frac{2}{\sqrt{y}}=4$ (A) $\frac{dy}{dx}=-\frac{3}{2}\left(\frac{y}{x}\right)^{3/2}$ (B) $\frac{dy}{dx}=\frac{3}{2}(xy)^{1/2}$

(C) $\frac{dy}{dx}=-\frac{2}{3}\left(\frac{x}{y}\right)^{3/2}$ (D) $\frac{dy}{dx}=\frac{2}{3}\left(\frac{x}{y}\right)^{3/2}$ (E) $\frac{dy}{dx}=\frac{3}{2}\left(\frac{y}{x}\right)^{3/2}$ (F) $\frac{dy}{dx}=\frac{2}{3}(xy)^{1/2}$

_____ 10. Find the equation of the tangent line to the graph of $y^2-xy-12=0$ at the point $(1,4)$.

(A) $3y=2x+10$ (B) $3y+2x=10$ (C) $y=4x$ (D) $7y=4x+24$ (E) $7y+4x=24$

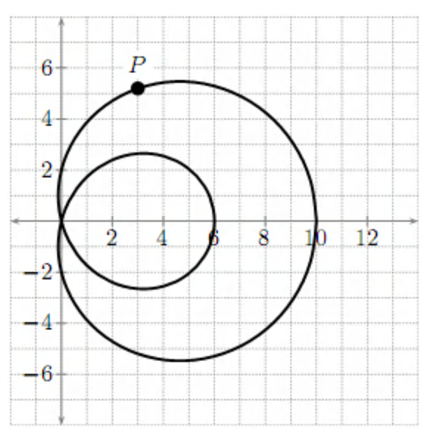
_____ 11. The slope of the tangent line to the graph of $x^3-2y^3+xy=0$ at the point $(1,1)$ is

(A) $-\frac{4}{5}$ (B) $\frac{3}{2}$ (C) $-\frac{5}{4}$ (D) $\frac{5}{4}$ (E) $\frac{4}{5}$ (F) $-\frac{2}{3}$

_____ 12. Determine $\frac{d^2y}{dx^2}$ when $4x^2+3y^2=4$

(A) $\frac{d^2y}{dx^2}=\frac{16}{9y^2}$ (B) $\frac{d^2y}{dx^2}=-\frac{16}{9y^2}$ (C) $\frac{d^2y}{dx^2}=-\frac{4}{9y^3}$ (D) $\frac{d^2y}{dx^2}=-\frac{16}{9y^3}$ (E) $\frac{d^2y}{dx^2}=\frac{16}{9y^3}$

_____ 13. The graph of the equation $(x^2+y^2-8x)^2=4(x^2+y^2)$ is shown at right. Find the equation of the tangent line to the graph at the point $(3,3\sqrt{3})$.



(A) $y=\frac{1}{\sqrt{3}}\left(\frac{5}{3}x-12\right)$ (B) $y=\frac{1}{\sqrt{3}}\left(\frac{5}{3}x+12\right)$ (C) $y=\frac{1}{\sqrt{3}}\left(\frac{3}{5}x+\frac{6\sqrt{3}}{5}\right)$

(D) $y=\frac{1}{\sqrt{3}}\left(\frac{3}{5}x-\frac{36}{5}\right)$ (E) $y=\frac{1}{\sqrt{3}}\left(\frac{3}{5}x+\frac{6}{5}\right)$

Related Rate of Change

Introduction

Change is an essential feature of the real world. In many situations a change in one quantity causes a change in another quantity or occurs together with a change in another quantity, with the result that the two rates of change are related. Consider the following problem:

> *An oil spill from a tanker spreads out in a circular pattern, centred at the tanker's position. If the edge of the oil spill is moving outwards at 2 m/s, find the rate of increase of the contaminated area when the radius is 500 m.*

This is an example of a "related-rates" problem. We are given the rate of change of radius with respect to time, $\frac{dr}{dt}$, and wish to calculate the rate of change of area with respect to time, $\frac{dA}{dt}$. Since area and radius are related by $A = \pi r^2$, and since both depend on time *t*, we can calculate $\frac{dA}{dt}$ by differentiating both sides of the equation $A = \pi r^2$ with respect to *t*. Differentiating both sides of an equation that relates two or more time-dependent variables forms the basis for the method of solving related-rates problems.

Strategy for Solving Related Rates Problems

1. **Read the problem carefully.**
2. **Draw and label a diagram if possible.**
3. **Introduce notation. Assign symbols to all quantities.**
4. **Express the given information and required rate in terms of derivatives.**
5. **Write an equation that relates the various quantities of the problem. You may need to know a formula for volume, area, surface area or Pythagoras.**
6. **Differentiate both sides of the equation with respect to time (t). You will probably use the chain rule.**
7. **Substitute the given information in the resulting equation and solve for the unknown rate.**

<u>RESIST TEMPTATION</u>

<u>Do not substitute numerical information until the end of the problem!</u>

BALLOON

Air is been pumped into a spherical balloon so that its volume increases at a rate of 80 cm^3/sec. How fast is the radius increasing when the diameter is 60 cm?

Relevant formula:	$V = \frac{4}{3}\pi r^3$
Known quantities:	
Unknown quantity:	
Differentiate wrt "t"	
Substitute	Answer: $\frac{1}{45\pi}$ cm/s

SNOWBALL

If a snowball melts so that its surface area decreases at a rate of 1 cm^2/sec, find the rate at which the diameter decreases when the diameter is 12 cm.

Relevant formulae:	
Known quantities:	
Unknown quantity:	
Differentiate wrt "t"	
substitute	**Answer**: $\frac{-1}{24\pi}$ cm/s

CIRCLE PROBLEM

A golf ball chipped into a water hazard creates a circular ripple effect. If the radius of the ripple is increasing at 0.8 m/s, how fast is the area changing when the radius is 6m?

Answer: $9.6\pi \frac{m^2}{s}$

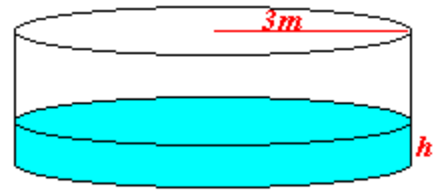
THE FALLING LADDER

A ladder is sliding down along a vertical wall. If the ladder is 10 meters long and the top is slipping at the constant rate of 10 m/s, how fast is the bottom of the ladder moving along the ground when the bottom is 6 meters from the wall?

Answer: $13\frac{1}{3}$ m/s

CYLINDER PROBLEM
A hose is filling a cylindrical swimming pool of radius 3m at a rate of 50 L / min. At what rate is the water level rising?

(Recall: 1 L / min = 0.001 m^3 / min)



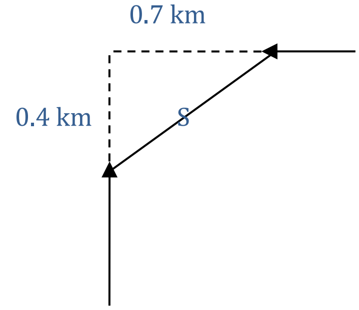
Answer: $\frac{1}{180\pi}$ m/min

CAR PROBLEM #1
Two cars start moving from the same point. Car A is traveling east at 60 m/h and car B is traveling south at 25 m/h. At what rate is the distance between them increasing 4 hours later?

Answer: 65 m/h

CAR PROBLEM #2

Car A is traveling north at 80 km/h and car B is traveling west at 110 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.4 km and car B is 0.7 km from the intersection?



Answer:-135.2 km/h

CAR PROBLEM #3

A train is 180km south of town **A**, travelling north at 40km/h. A car leaves town **A** driving west at 80km/h. At what rate is the distance between them changing 2 hours later?

SHIP PROBLEM

At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between ships changing at 4:00pm?

Answer: 55.4 km/h

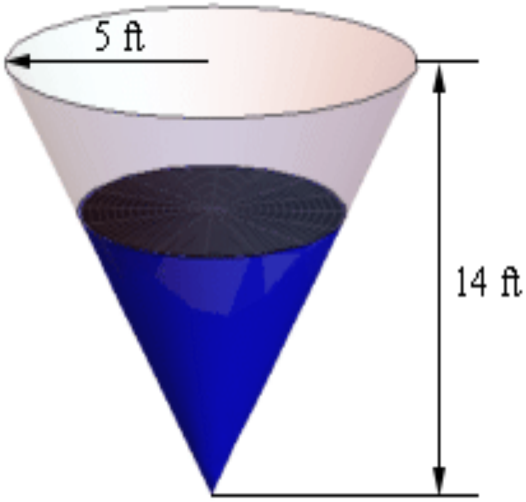
CONE PROBLEM #1

A tank of water in the shape of a cone is leaking water at a constant rate of $2m^3/h$. The base radius of the tank is 5 m and the height of the tank is 14 m.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 m?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 m?

Diagram:



$$V = \frac{1}{3}\pi r^2 h$$

Answer: a) $\dfrac{-98}{225\pi}$ m/h b) $\dfrac{-7}{45\pi}$ m/h

CONE PROBLEM #2

Gravel is being dumped from a conveyor belt at a rate of 25 cubic feet per minute and it forms a pile in the shape of a cone whose base diameter is twice the height. How fast is the height increasing when the pile is 12 feet high?

CONE PROBLEM #3

Sand pouring from a conveyor belt forms a conical pile, the radius of which is $\frac{3}{4}$ of the height. If the sand is piling up at a constant rate of $\frac{1}{2}$ m^3/min, at what rate is the height of the pile growing 3 min after the pouring starts.

Answer. 0.152 m/min

CONE PROBLEM #4

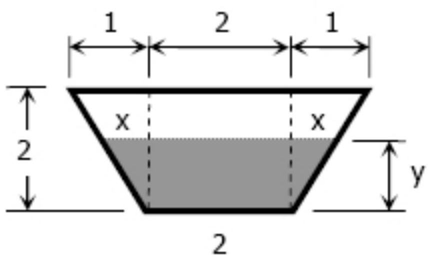
A cone with height of 30cm and a diameter of 20cm contains water. Water is leaking out of the conical cup at a constant rate such that the depth of the water, ***h,*** is decreasing at 2 cm/min when the depth of the water is 20 cm. The water is then collecting in a box with a square base 10 cm per side. At what rate is the depth of the water, ***y***, increasing in the box?

Answer: $\mathbf{\frac{8\pi}{9}\ cm/min}$

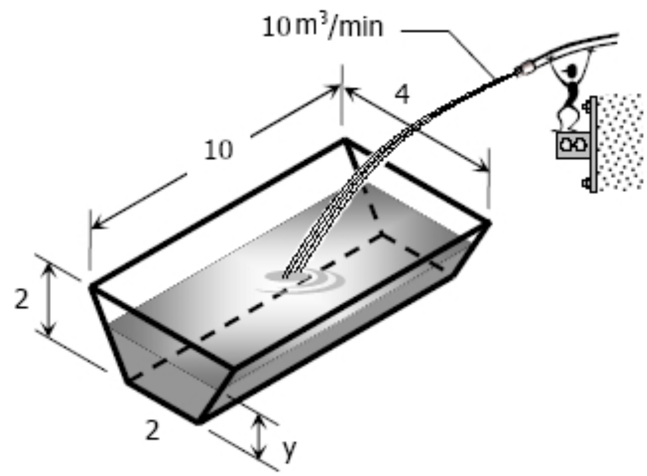
1. Water is being pumped into a trough that is 4.5m long and has a cross section in the shape of an equilateral triangle 1.5m on a side. If the rate of inflow is 2 cubic meters per minute how fast is the water level rising when the water is 0.5m deep?

2. The cross section of a 10-meter trough is an isosceles trapezoid with a 2-meter lower base, a 4-meter upper base, and an altitude of 2 meters.

a) Write an expression for the volume of water in the trough as a function of y.



Cross Section



b) Water is running into the trough at a rate of 10 cubic meters per minute. How fast is the water level rising when the water is 0.5 meter deep?

3. A trough filled with water is 2 m long and has a cross section in the shape of an isosceles trapezoid 30 cm wide at the bottom, 60 cm wide at the top, and a height of 50 cm., if the trough leaks water at the rate of 2000 cm^3/min, how fast is the water level falling when the water is 20 cm deep?

4. A trough is 12 feet long and 3 feet across the top. Its ends are isosceles triangles with altitudes of 3 feet

   a) If water is being pumped into the trough at a rate of 2 cubic feet per minute, how fast is the water level rising when the depth is 1 foot?

   b) If the water is rising at a rate of 3/8 inch per minute when h = 2, determine the rate at which the water is being pumped into the trough

RELATED RATES QUESTIONS

Part I

1. Two cars start moving from the same point. One travels south at 60 km/h and the other travels west at 25 km/h. At what rate is the distance between the cars increase two hours later?[65 km/h]

2. Car A is traveling west at 50 km/h and car B is traveling north at 60 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 km from the intersection and car B is 0.4 km from the intersection?[-78km/h]

3. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides from the wall at a rate of 0.2m/s. How long is the ladder?[5 m]

4. How fast is the area of a circle increasing when the circle's radius is 2 m and growing at a rate of 5 m/min? [20π m^2/min]

5. Gas is escaping from a spherical weather balloon at a rate of 50cm^3/minute. At what rate is the surface area (S) shrinking when the radius is 15 centimeters? [$-\frac{20}{3}$ cm^2/min]

6. A square is expanding so that its area increases at 10 cm^2/min. How fast is the side length increasing when the area is 52 cm^2? [$\frac{5\sqrt{13}}{26}$ cm/min]

7. Josh is 200 m directly north of a buoy, and swims towards it at a rate of 1.2 m/s. Kim is 240 m directly east of the same buoy and swims towards it at a rate of 1.4 m/s. Determine the rate at which the distance between the two swimmers is changing 100 seconds after they start. [-1.84 m/s]

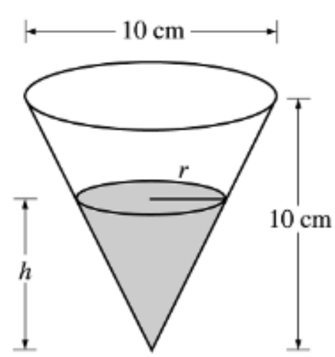
Part II

1. Water is pouring into a conical tank at the rate of 8 cubic meter per minute. If the height of the tank is 12 m and the radius of its circular opening is 6 m, how fast is the water level rising when the water is 4 m deep?

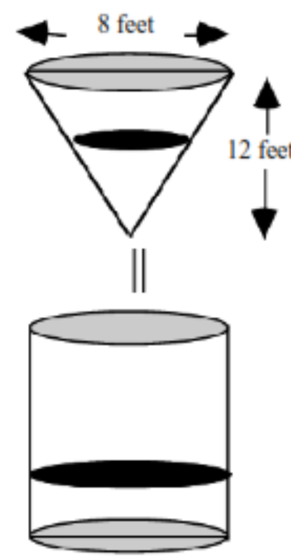
2. Sand is falling off a conveyor belt at a rate of 12 cubic meter per minute into a conical pile. The diameter of the pile is four times the height. At what rate is the height of the pile changing when the pile is 10 m high?

3. A water tank has the shape of an inverted circular cone with a base diameter of 8 m and a height of 12 m.
   a) If the tank is being filled with water at the rate of 5 m^3/min, at what rate is the water level increasing when the water is 5 m deep?
   b) If the water tank is full of water and being drained at the rate of 7 m^3/min, at what rate is the water level decreasing when the water is 7 m deep?

4. A container has the shape of an open right circular cone, as shown in the figure below. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/h.
   a) Find the volume V of water in the container when h = 5 cm. Indicate units of measure.
   b) Find the rate of change of the volume of water in the container, with respect to time, when h= 5 cm. Indicate units of measure.
   c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?



5. As shown in the figure below, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h - 12)$ feet per minute.
   a) Write an expression for the volume of water in the conical tank as a function of h.
   b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.
   c) Let y be the depth, in feet, of water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.



Part II -Answers

1. $\frac{2}{\pi}$ m/min   2. $\frac{3}{100\pi}$ m/min

3. a) increasing at a rate of 0.573 m/min
   b) decreasing at a rate of 0.409 m/min

4. a) $\frac{125\pi}{12}\text{cm}^3$ b) $-\frac{15\pi}{8}\text{cm}^3/\text{hr}$ c) $k = \frac{dh}{dt} = -\frac{3}{10}$

5. a) $V = \frac{\pi}{27}h^3$ b) $\frac{dV}{dt} = -9\pi\ \text{ft}^3/\text{min}$ c) $\frac{dy}{dt} = \frac{9}{400}$ ft/min

Relax.....Review

1. Differentiate the following:

a) $m(t)=\frac{\pi}{3}t^3-3t^{-5}+4\pi^2$

b) $f(x)=\left(1+x^{\frac{3}{4}}\right)\left(\sqrt{x+\sqrt{x}}\right)$ (do not simplify)

c) $g(x)=\frac{x^3+4}{x^3-3x+1}$

d) $y=\frac{(3x^2-1)^{-4}}{(x^3-2x)^{-5}}$

2. If $g(x)=\frac{1}{2x-4}+\sqrt{x}$, find $g'''(4)$.

3. For what value(s) of *k* will the line $2x-3y+k=0$ be normal to $y=\sqrt{3x^2+4}$?

4. Find the rate of change for $s(t)=\left(\frac{t-\pi}{t-10\pi}\right)^{\frac{1}{3}}$ at $t=2\pi$. Leave final answer in terms of π.

5. Two tangents are drawn from the point (2,6) to the graph of $y=-x^2-5x+4$. Determine the coordinates of the point(s) where the tangents touch the graph.

6. For what values of *a* and *b* will the parabola $y=x^2+ax+b$ be tangent to the curve $y=x^3+5x$ at the point x=1?

7. A 1500-L tank leaks water so that the volume of water, in litres, remaining after *t* days, $0\le t\le 15$, is represented by $V(t)=1500\left(1-\frac{t}{15}\right)^2$. How rapidly is the water leaking when the tank is $\frac{1}{9}$ full? Round final answers to 2 decimal places.

8. Find the values of *x* so that the tangent to $f(x)=\frac{3}{\sqrt[3]{x}}$ is parallel to the line *x* + 16*y* + 3 =0.

9. Find **a** and **b** so that the line y = - ax +4 is tangent to the graph of y= ax³ + bx at $x=1$.

10. Find the constant value(s) of ***k*** such that the equation of tangent to the curve $f(x)=\sqrt{1-kx^2}$ at $x=1$ is parallel to the line $3x-2y+1=0$.

11. Two lines drawn from point $A\left(0,\frac{7}{4}\right)$ are tangent to the parabola $y=1-x^2$ at P and Q. Find the area of triangle APQ.

12. Let f be a function given by $f(x)=\frac{ax^2+b}{x+c}$ and that has the following properties

$$\lim_{x\to -1^-} f(x)=\infty,\qquad f'(0)=2,\qquad f''(0)=-2.$$ Determine the values of a, b and c

13. Let $f(x)=\sqrt{ax^2+b}$. Find values of a and b such that the linear equation $7x+2y=5$ is tangent to $f(x)$ at $x=-1$.

14. Find the area of the triangle determined by the coordinate axes and the tangent to the curve *xy* = 1 at *x* = 1.

15. Consider the curve $y = a\sqrt{x} + \dfrac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

16. The equation of the tangent to $y = 2x^2 - 1$ at the point where $x = 1$, is $4ax - y = 2b^2 + 1$. Find the values of a and b.

17. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ so that the line $y = -4x + 1$ is tangent to the graph of $y = \dfrac{a}{x} + \dfrac{b}{x+1}$ at $x = 1$.

18. The curve $y = 2x^3 + ax + b$ has a tangent with slope 10 at the point (-2, 33). Find the values of a and b.

19. The position of an object moving along a straight line is described by the function $s(t) = -t^3 + 4t^2 - 10 \quad for \ t \geq 0.$
    (a) Is the object moving away from or towards its initial position when t = 3?
    (b) Is the object speeding up or slowing down when t = 3?

20. A position function of an object is given by: $s(t) = t^3 - 6t^2 + 8t \quad , t \geq 0$
    (a) Determine the velocity function for the object.
    (b) Identify the point(s) where the object is at rest.
    (c) Identify the point(s) where the acceleration is zero.
    (d) Determine the equation of the acceleration function.
    (e) For which intervals is the acceleration negative? Positive?
    (f) Determine the intervals for which the object is speeding up and slowing down.

21. A north-south highway intersects an east-west highway at a point P. An automobile crosses P at 10:00 AM, travelling east at a constant speed of 20 km / hr. At the same instant, another automobile is 2km north of P , travelling south at 50km/hr. Find the time at which they are closest to each other, and approximate the minimum distance between the automobile.

22. At noon a car is driving west at 55 km/h. At the same time, 15 km due north another car is driving south at 85 km/h. At what rate the distance between two cars changing 4 hours later?

23. How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s ?

24. A water tank is in the shape of an inverted right circular cone with top radius 10 m and depth 8 m. Water is flowing in at a rate of 0.1 m^3 /min. How fast is the depth of water in the tank increasing 3 minutes later?

25. A ladder 25 m long is leaning against a house. The base of the ladder is pulled away from the wall at a rate of 2 meter per second. How fast is the top of the ladder moving down the wall when the base of the ladder is 12 meter from the wall? Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 m from the wall.