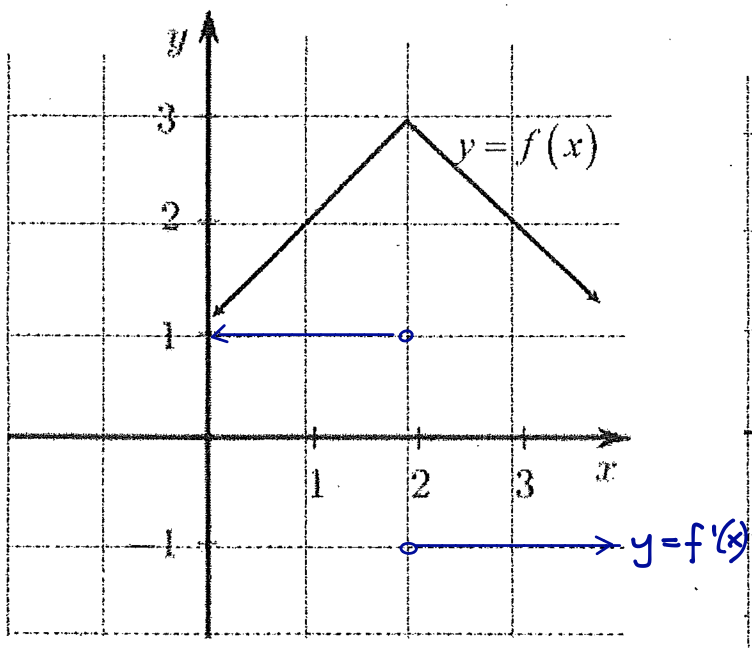
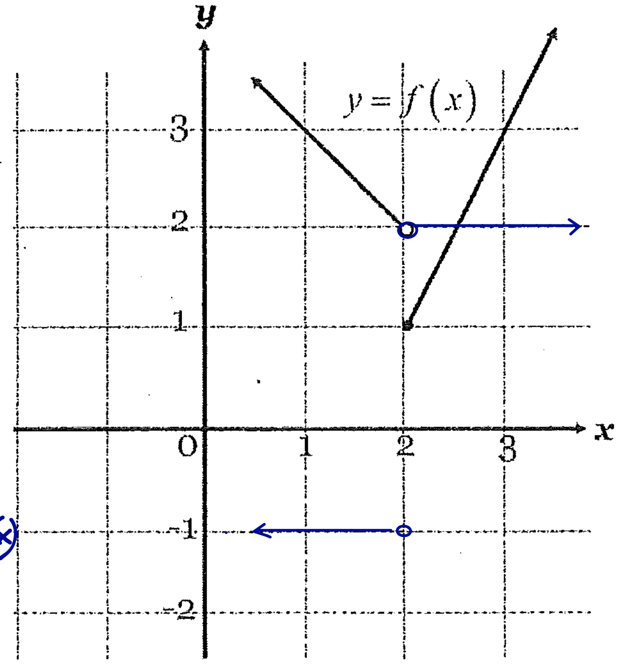
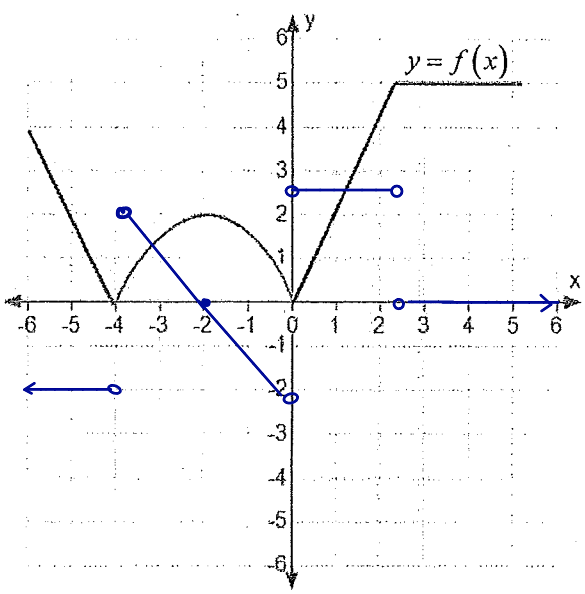
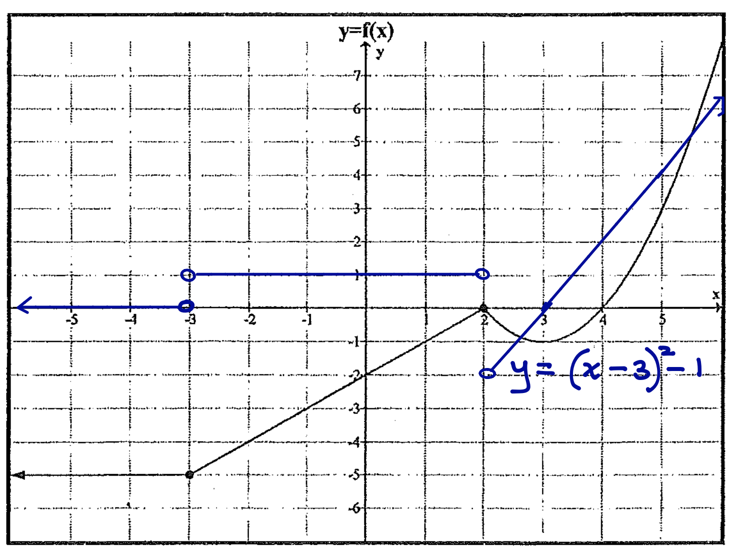
Example: graph of function $f(x)$ is given. Sketch the graph of $f'(x)$ on the same grid.









Power, Sum & Difference Rules

Recall: What is a Derivative? The derivative of the function *f* at the number *a* is the slope of the curve $y = f(x)$ at $x = a$. The symbol for the derivative of *f* at the number *a* is $f'(a)$. Putting this together with the answers to our two questions above, we get

> *The derivative of f at the number a is given by*
>
> $$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

This leads to the definition of the derivative function

> *The derivative of f(x) with respect to x is the function $f'(x)$, where*
>
> $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Other Notation for Derivatives

Other notations for the derivative of the function $y = f(x)$ are $f'(x)$, y', and $\frac{dy}{dx}$

More About Derivatives

Since the derivative $f'(a)$ can be interpreted as the slope of the tangent at $(a, f(a))$, it follows that the derivative $f'(a)$ can also be considered the instantaneous rate of change of $f(x)$ with respect to *x* when $x = a$.

The Power Rule

> If $f(x) = x^n, n \in R$, then $f'(x) = nx^{n-1}$.
>
> Stated in Leibniz notation, if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$.

The Constant Multiple Rule

> If $f(x) = kg(x), k \in R$, then $f'(x) = kg'(x)$.
>
> Stated in Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$.

The Sum Rule[1]

> If $f(x) = p(x) + q(x)$, where $p(x)$ and $q(x)$ are both differentiable functions, then $f'(x) = p'(x) + q'(x)$.
>
> Stated in Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$.

[1] A corollary of the constant multiple rule and the sum rule is that if $f(x)=p(x) - q(x)$, then $f'(x)=p'(x)-q'(x)$

Examples

1. Determine the derivatives of each of the following

a) $f(x) = 4x^5$

$f'(x) = 20x^4$

b) $g(x) = 11x^{\frac{5}{2}}$

$g'(x) = \frac{55}{2}x^{\frac{3}{2}}$

c) $h(x) = 4x^3 - 3\sqrt{x}$

$h'(x) = 12x^2 - \frac{3}{2}x^{-\frac{1}{2}}$

$= 12x^2 - \frac{3}{2\sqrt{x}}$

d) $k(x) = (5x-3)^2$

$k(x) = 25x^2 - 30x + 9$

$k'(x) = 50x - 30$

e) $m(x) = \dfrac{4x^5 - \pi x^7}{5x^3}$

$m(x) = \frac{4}{5}x^2 - \frac{\pi}{5}x^4$

$m'(x) = \frac{8}{5}x - \frac{4\pi}{5}x^3$

f) $n(x) = 7x^4 + \sqrt{x} - 3x^{\frac{3}{2}} - \dfrac{2}{x^4} - 99$

$n(x) = 7x^4 + x^{\frac{1}{2}} - 3x^{\frac{3}{2}} - 2x^{-4} - 99$

$n'(x) = 28x^3 + \frac{1}{2}x^{-\frac{1}{2}} - \frac{9}{2}x^{\frac{1}{2}} + 8x^{-5}$

$n'(x) = 28x^3 + \frac{1}{2\sqrt{x}} - \frac{9}{2}x^{\frac{1}{2}} + \frac{8}{x^5}$

2. Determine the equation of the tangent to the graph of $y = x^3 + 2x^2 - 4x + 1$ at $x = 4$

$y = x^3 + 2x^2 - 4x + 1$

$m_t = y' = 3x^2 + 4x - 4$

At $x = 4$, $m_t = 3(4)^2 + 4(4) - 4$

$m_t = 60$

At $x = 4$, $y = 4^3 + 2(4)^2 - 4(4) + 1$

$= 64 + 32 - 16 + 1$

$= 81$

Equation of tangent line is

$y - y_1 = m(x - x_1)$

$y - 81 = 60(x - 4)$

$y = 60x - 240 + 81$

$y = 60x - 159$ (slope y-intercept form)

or $60x - y - 159 = 0$ (standard form $Ax + By + C = 0$, A, B, and C are integers)

3. A cubic polynomial function, $f(x)=ax^3+bx^2+cx+d$,is given such $f'(0)=0$, $f'(1)=5$, $f'(2)=16$,find $f'(3)$.

$f'(x)=3ax^2+2bx+c$

$f'(0)=0$

$0=3a(0)^2+2b(0)+c$

$\boxed{c=0}$

$f'(1)=5$

$5=3a(1)^2+2b(1)+0$

$5=3a+2b$ —— ①

$f'(2)=16$

$16=3a(2)^2+2b(2)+0$

$16=12a+4b$

or $4=3a+b$ —— ②

(or $b=4-3a$)

② − ① $\boxed{b=1}$

Sub $b=1$ in ②

$4=3a+(1)$

$3=3a$

$\boxed{a=1}$

$f'(x)=3x^2+2x$

$f'(3)=3(3)^2+2(3)$

$\boxed{f'(3)=33}$

4. Determine the point(s) where the tangent to the curve $f(x) = x^3 - 6x^2 + 7$:

a) Has a slope of -9

$m_t=-9$

$f'(x)=3x^2-12x$

$-9=3x^2-12x$

$3x^2-12x+9=0$

$3(x^2-4x+3)=0$

$3(x-3)(x-1)=0$

$x=3$ or $x=1$

$f(3)=3^3-6(3)^2+7$

$\boxed{y=-20}$

$f(1)=1^3-6(1)^2+7$

$\boxed{y=2}$

The points are $(3,-20)$ and $(1,2)$

b) Is horizontal

$m_t=0$

$f'(x)=3x^2-12x$

$0=3x(x-4)$

$x=0$ or $x=4$

$f(0)=7$

$\boxed{y=7}$

$f(4)=4^3-6(4)^2+7$

$\boxed{y=-25}$

The points are $(0,7)$ and $(4,-25)$

5. Find the values of x so that the tangent to $f(x) = \frac{3}{\sqrt[3]{x}}$ is parallel to the line x + 16y + 3 =0

$16y = -x - 3$

$y = \frac{-x}{16} - \frac{3}{16}$

$m_t = -\frac{1}{16}$

$f(x) = 3x^{-\frac{1}{3}}$

$f'(x) = -x^{-\frac{4}{3}}$

$-\frac{1}{16} = -\frac{1}{x^{\frac{4}{3}}}$

$x^{\frac{4}{3}} = 16$

$x^4 = 4096$

$\boxed{x = \pm 8}$

6. Find the values for *a* and *b* so that *f(x)* is **differentiable** for all *x*.

$$f(x) = \begin{cases} -x^3 + 2x^2 + 4 & x \le 1 \\ ax + b & x > 1 \end{cases}$$

$f(x)$ is differentiable $\Leftrightarrow$ $f(x)$ is continuous

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$\lim_{x \to 1^-} -x^3 + 2x^2 + 4 = \lim_{x \to 1^+} ax + b$

$-1 + 2 + 4 = a + b$

$a + b = 5$ — (1)

$f(x)$ is differentiable at $x = a$ $\Leftrightarrow$ $f'(a^-) = f'(a^+)$

$f'(1^-) = f'(1^+)$

$-3x^2 + 4x = a$

$-3 + 4 = a$

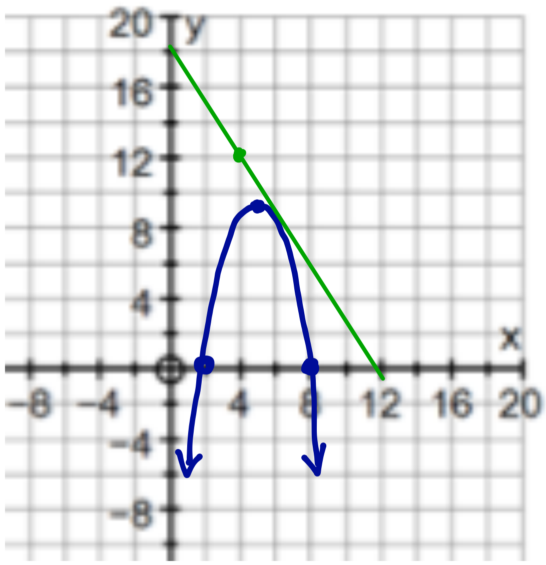
$\boxed{a = 1}$

Sub in (1)

$1 + b = 5$

$\boxed{b = 4}$

Ex. Given the parabola $y = 10x - x^2 - 16$. Find the equations of the two tangents that pass through the point P(4, 12). (Hint: Graph the function)



Sketch the function:

Vertex form: $y = -(x-5)^2 + 9$

Vertex: $(5, 9)$ Zeros: 2, 8

1) Since (4, 12) does not lie on the parabola, let Q (a, f(a)) be one point of tangency.

$$\frac{dy}{dx} = 10 - 2x$$

$$\left.\frac{dy}{dx}\right|_{x=a} = 10 - 2a$$

2) Find the slope through P and Q:

$$m_{PQ} = \frac{f(a) - 12}{a - 4}$$

$$= \frac{(10a - a^2 - 16) - 12}{a - 4}$$

$$= \frac{-a^2 + 10a - 28}{a - 4}$$

3) Equate $\left.\frac{dy}{dx}\right|_{x=a}$ and m_{PQ}

$$\frac{-a^2 + 10a - 28}{a - 4} = 10 - 2a$$

$$-a^2 + 10a - 28 = (10 - 2a)(a - 4)$$

$$-a^2 + 10a - 28 = 10a - 2a^2 - 40 + 8a$$

$$a^2 - 8a + 12 = 0$$

$$(a - 6)(a - 2) = 0$$

$$\therefore a = 6 \text{ or } a = 2$$

4) First tangent line from P at $x = 2$:

At $x = 2$,

$y = 10(2) - (2)^2 - 16$

$= 0$

$\left.\frac{dy}{dx}\right|_{x=2} = 10 - 2(2)$

$= 6$

$\therefore y - (0) = (6)[x - (2)]$

$y = 6x - 12$

Second tangent line from P at $x = 6$:

At $x = 6$,

$y = 10(6) - (6)^2 - 16$

$= 8$

$\left.\frac{dy}{dx}\right|_{x=6} = 10 - 2(6)$

$= -2$

$\therefore y - (8) = (-2)[x - (6)]$

$y = -2x + 20$

$\therefore$ Two tangent lines are $y = 6x - 12$ and $y = -2x + 20$

Power Rule Solutions

Challenging Questions-Solutions

1. Determine the derivatives of each of the following

a) $f(x) = -\frac{3}{4}\sqrt[4]{x^5} - \frac{4}{3\sqrt{x^3}} + \pi^2 x^3 - \frac{7}{3}$

Done in class

b) $g(x) = 4\sqrt[4]{x^3}\left(\pi\sqrt[5]{x} - 2^3\pi\right)$

$$g(x) = 4x^{\frac{3}{4}}\left(\pi x^{\frac{1}{5}} - 2^3\pi\right)$$
$$= 4\pi x^{\frac{19}{20}} - 32\pi x^{\frac{3}{4}}$$
$$g'(x) = \frac{19\pi}{5}x^{\frac{-1}{20}} - 24\pi x^{\frac{-1}{4}}$$
$$= \frac{19\pi}{5x^{\frac{1}{20}}} - \frac{24\pi}{x^{\frac{1}{4}}}$$

c) $g(x) = \frac{3x^4 + 2\pi x^3 - 5\sqrt[3]{x}}{4x^2}$

$$g(x) = \frac{3}{4}x^2 + \frac{\pi}{2}x - \frac{5}{4}x^{-\frac{5}{3}}$$
$$g'(x) = \frac{3}{2}x + \frac{\pi}{2} + \frac{25}{12}x^{-\frac{8}{3}}$$
$$g'(x) = \frac{3}{4}x^2 + \frac{\pi}{2}x - \frac{25}{12x^{\frac{8}{3}}}$$

2. Find the slope of the tangents to $f(x) = x^2 - x + 4$ such that they pass through an exterior point P(3,2).

$f(x) = x^2 - x + 4$

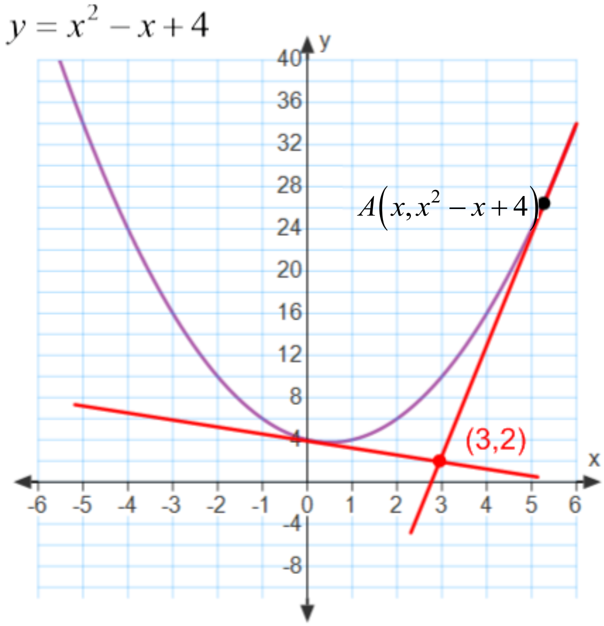
$m_T = f'(x) = 2x - 1$

Slope of line passes through points $A(x, x^2 - x + 4)$ and $P(3,2)$ is:

$$m_T = \frac{x^2 - x + 4 - 2}{x - 3} \rightarrow m_T = \frac{x^2 - x + 2}{x - 3}$$
$$\frac{x^2 - x + 2}{x - 3} = 2x - 1$$
$$x^2 - x + 2 = (2x - 1)(x - 3)$$
$$x^2 - x + 2 = 2x^2 - 7x + 3$$
$$x^2 - 6x + 1 = 0$$
$$x = 3 \pm 2\sqrt{2}$$

Slope of tangent lines are: $m_T = 5 \pm 4\sqrt{2}$

$y = x^2 - x + 4$

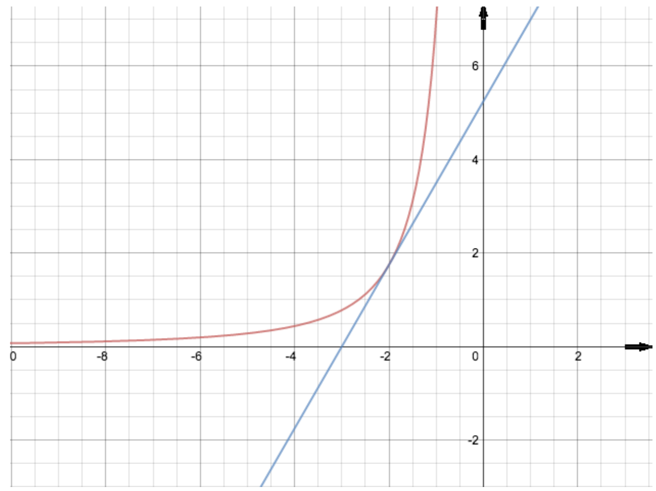


Power Rule Solutions

3. Determine the value of a, given that the line ax – 4y + 21 = 0 is tangent to the graph of $y = \frac{a}{x^2}$ at x = -2.

At $x = -2$ function $y = \frac{a}{x^2}$ and its tangent line $ax - 4y + 21$ have the same y-value,therefore:

$$a(-2) - 4\left(\frac{a}{4}\right) + 21 = 0$$
$$-3a = -21$$
$$a = 7$$

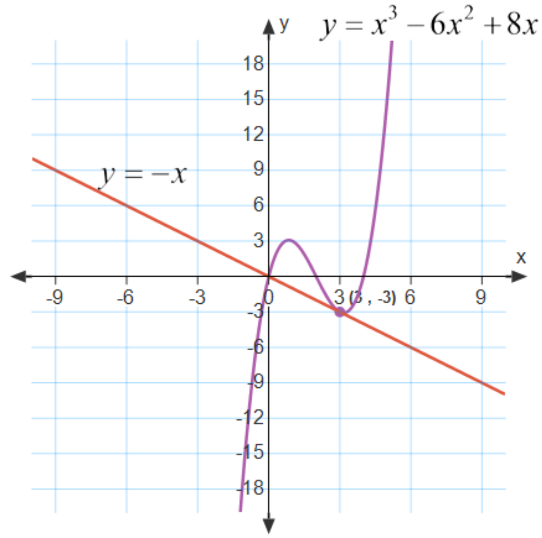


4. The tangent to the cubic function y = x^3 – 6x^2 + 8x at point A (3,-3) intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.

$$y = x^3 - 6x^2 + 8x$$
$$y' = 3x^2 - 12x + 8$$
$$m_T = 3(3)^2 - 12(3) + 8$$
$$m_T = -1$$

Equation of tangent line: $y = -x + b \leftarrow (3,-3)$

$$-3 = -3 + b$$
$$b = 0$$
$$\therefore y = -x$$



$y = x^3 - 6x^2 + 8x$ $\quad \Rightarrow -x = x^3 - 6x^2 + 8x$

$y = -x$ $\quad x^3 - 6x^2 + 9x = 0$

$$x(x-3)^2 = 0$$

$\swarrow \quad \searrow$

$x = 0 \qquad x = 3$

$\Downarrow \qquad \Downarrow$

$y = 0 \qquad y = -3$

Power Rule Solutions

5. Find the equations of the tangent lines to the parabola $y = x^2 + x$ that pass through the point (2, -3). Sketch the curve and tangents.

$f(x) = x^2 + x$

$m_T = f'(x) = 2x + 1$

Slop of tangent line passes through $A(x, x^2 + x)$

and P(2,-3) is:

$m_T = \dfrac{x^2 + x + 3}{x - 2}$

$2x + 1 = \dfrac{x^2 + x + 3}{x - 2}$

$(2x+1)(x-2) = x^2 + x + 3$

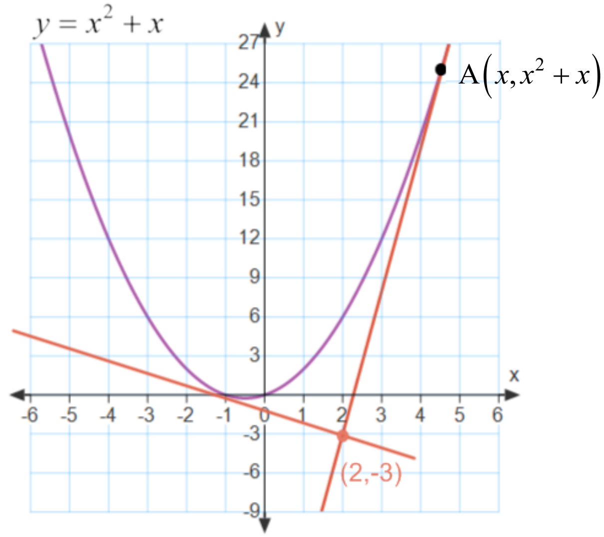
$2x^2 - 3x - 2 = x^2 + x + 3$

$x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$

$x = -1$ & $x = 5$

equation of tangent lines are: $y = -x - 1$ & $y = 11x - 25$



6. Two perpendicular lines which intersect at C are also the tangent lines to parabola $f(x) = \dfrac{-1}{10}(x-6)^2 - 3$ at the points A and B when $y = -5.5$. Find C.

$f(x) = \dfrac{-1}{10}(x-6)^2 - 3$

$y = -5.5$

$\Rightarrow$

$\dfrac{-1}{10}(x-6)^2 - 3 = -5.5$

$(x-6)^2 - 30 = 55$

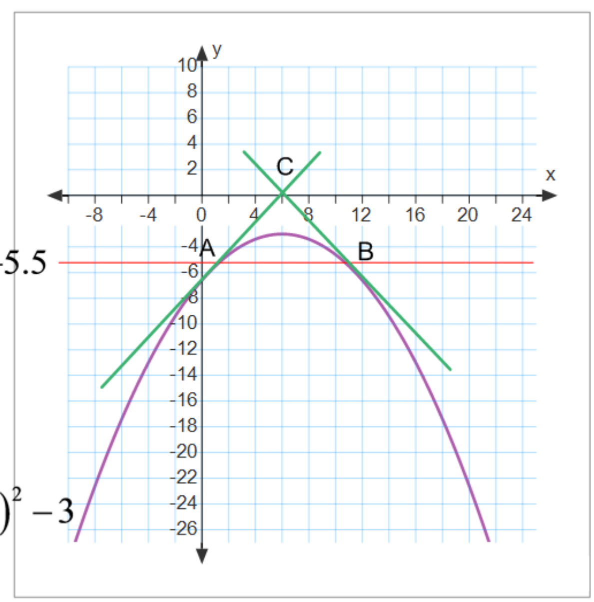
$(x-6)^2 = 25$

$x - 6 = \pm 5 \Rightarrow x = 11$ or $x = 1$ $y = -5.5$

$f(x) = \dfrac{-1}{10}(x^2 - 12x + 36) - 3$

$f'(x) = \dfrac{-1}{10}(2x - 12)$

$= \dfrac{-1}{5}(x - 6)$



$m_T = \frac{-1}{5}(11-6)$ & $m_T = \frac{-1}{5}(1-6)$

$= -1$ $= 1$

$y = -x + b$ $y = x + b$

$-5.5 = -11 + b$ $-5.5 = 1 + b$

$b = 5.5$ $b = -6.5$

$y = -x + 5.5$ $y = x - 6.5$

$$\begin{cases} y = -x + 5.5 \\ y = x - 6.5 \end{cases} \Rightarrow y = -0.5\ ,\ x = 6$$

$$\therefore C(6, -0.5)$$

WARM -UP: DERIVATIVE RULES

1. Differentiate the following. Express the answers with positive exponents.

a) $f(x)=x^{-4}-\sqrt{2x}-\frac{5x}{\sqrt{x}}+8^2$

$f(x)=x^{-4}-\sqrt{2}x^{\frac{1}{2}}-5x^{\frac{1}{2}}+64$

$f'(x)=-4x^{-5}-\frac{\sqrt{2}}{2}x^{-\frac{1}{2}}-\frac{5}{2}x^{-\frac{1}{2}}$

$f'(x)=\frac{-4}{x^5}-\frac{\sqrt{2}}{2\sqrt{x}}-\frac{5}{2\sqrt{x}}$

$f'(x)=-\frac{4}{x^5}-\frac{\sqrt{2}+5}{2\sqrt{x}}$

b) $g(x)=x^{-2}\left(x^{-1}+1\right)^2$

$g(x)=x^{-2}(x^{-2}+2x^{-1}+1)$

$g(x)=x^{-4}+2x^{-3}+x^{-2}$

$g'(x)=\frac{-4}{x^5}-\frac{6}{x^4}-\frac{2}{x^3}$

2 The equation of the tangent to $y=ax^3+kx+1$ is $y=4x+k$ at $x=1$. Find the values of a and k. At what point does this tangent intersect the curve again?

At $x=1$, $y=4+k$ —— ①

Also, $y=a(1)^3+k(1)+1$

$y=a+k+1$ —— ②

① = ② at point of tangency

$4+k=a+k+1$

$\boxed{a=3}$

$y=4x+k$

$m_t=4$

$y=ax^3+kx+1$

$y'=3ax^2+k$

At $x=1$ $4=3a(1)+k$

$4=3a+k$ —— ③

Sub $a=3$ in ③

$4=3(3)+k$

$\boxed{k=-5}$

$y=3x^3-5x+1$

and $y=4x-5$

At point of intersection

$4x-5=3x^3-5x+1$

$3x^3-9x+6=0$

$3(x^3-3x+2)=0$

Let $f(x)=x^3-3x+2$

$f(1)=0 \Rightarrow x-1$ is a factor of $f(x)$

-1	1	0	-3	2
		-1	-1	2
	1	1	-2	0

$(x-1)(x^2+x-2)=0$

$(x-1)(x+2)(x-1)=0$

$(x-1)^2(x+2)=0$

$x=1$ or $x=-2$

At $x=-2$, $y=4(-2)-5$

$\boxed{y=-13}$

The tangent will intersect the curve again at $(-2,-13)$

The Product Rule

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$.

Restated in Leibniz notation,

If u and v are functions of x, $\dfrac{d}{dx}(uv) = \dfrac{du}{dx}v + u\dfrac{dv}{dx}$

Proof of the Product Rule

Suppose $p(x) = f(x)g(x)$. Then

$$p'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{(x+h)-x}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) + 0 - f(x)g(x)}{h}$$

Just like multiplying by 1 is a powerful tool in math, so is adding 0
In this case, $0 = -f(x)g(x+h) + f(x)g(x+h)$
Who said mathematicians aren't creative? ☺

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

Next, we'll factor out $g(x+h)$ from the first two terms of the numerator, and we'll factor out $f(x)$ from the last two terms of the numerator

$$= \lim_{h\to 0}\left\{\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h) + f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= f'(x)g(x) + f(x)g'(x)$$

Examples

1. Differentiate $h(x) = (x^3 - 2x)(3x^4 + 2x + 8)$ using the product rule

$$h'(x) = (3x^2-2)(3x^4+2x+8) + (x^3-2x)(12x^3+2)$$

$$= 9x^6 + 6x^3 + 24x^2 - 6x^4 - 4x - 16 + 12x^6 - 24x^4 + 2x^3 - 4x$$

$$= 21x^6 - 30x^4 + 8x^3 + 24x^2 - 8x - 16$$

2. Find the value of $f'(-1)$ for the function $f(x)=(3x^4-12x^2+4x-9)(6x^7-4x^4+18)$

$f'(x)=(12x^3-24x+4)(6x^7-4x^4+18)+(3x^4-12x^2+4x-9)(42x^6-16x^3)$

$f'(-1)=[12(-1)^3-24(-1)+4][6(-1)^7-4(-1)^4+18]+[3(-1)^4-12(-1)^2+4(-1)-9][42(-1)^6-16(-1)^3]$

$=(-12+24+4)(-6-4+18)+(42+16)(3-12-4-9)$

$=(16)(8)+(58)(-22)$

$=-1148$

Find an expression for $p'(x)$ if $p(x)=f(x)g(x)h(x)$

$p(x)=f(x)g(x)h(x)$

$=[f(x)g(x)]h(x)$

$p'(x)=[f(x)g(x)]'h(x)+[f(x)g(x)]h'(x)$

$$p'(x)=f'(x)g(x)h(x)+f(x)g'(x)h(x)+f(x)g(x)h'(x)$$

This is called the ***extended product rule*** for three functions

3. Differentiate the rational function $f(x)=x(2x+5)(x-1)$ by using the extended product rule.

$f'(x)=1(2x+5)(x-1)+x(2)(x-1)+x(2x+5)(1)$

$=2x^2-2x+5x-5+2x^2-2x+2x^2+5x$

$=6x^2+6x-5$

4. If $g(x)=x^2f(x)$, $f(2)=-2$ and $g'(2)=8$, then determine $f'(2)$.

$g'(x)=2xf(x)+x^2f'(x)$

$g'(2)=2(2)f(2)+(2)^2f'(2)$

$8=4(-2)+4f'(2)$

$\frac{8+8}{4}=f'(2)$

$\boxed{f'(2)=4}$

<u>The Power of a Function Rule for Integers</u>

If u is a function of x, and n is an integer, then $\frac{d}{dx}(u^n)=nu^{n-1}\frac{du}{dx}$

In function notation, if $f(x)=[g(x)]^n$, then $f'(x)=n[g(x)]^{n-1}g'(x)$

We will prove a more general statement of this (the Chain Rule) in section

Examples

1. Determine $h'(x)$ where $h(x)=(4x^2-3x+1)^7$. Then, evaluate $h'(1)$.

$h'(x) = 7(4x^2-3x+1)^6(8x-3)$

$h'(1) = 7(4-3+1)^6(8(1)-3)$
$= 7(2)^6(5)$
$= 2240$

2. Find the derivative of $g(x)=\left(3x^2-5\right)^6\left(2x^3+1\right)^4$.

$g'(x) = 6(3x^2-5)^5(6x)(2x^3+1)^4 + (3x^2-5)^6(4)(2x^3+1)^3(6x^2)$
$= 36x(3x^2-5)^5(2x^3+1)^4 + 24x^2(2x^3+1)^3(3x^2-5)^6$ ⇐ Factor
$= 12x(3x^2-5)^5(2x^3+1)^3[3(2x^3+1)+2x(3x^2-5)]$
$= 12x(3x^2-5)^5(2x^3+1)^3(6x^3+3+6x^3-10x)$
$= 12x(3x^2-5)^5(2x^3+1)^3(12x^3-10x+3)$

3. Differentiate the following. Write the final answer with positive exponents.

$$h(x)=\frac{(2x-1)^2}{(3x+2)^3}$$

$h(x) = (2x-1)^2(3x+2)^{-3}$
$h'(x) = 2(2x-1)(2)(3x+2)^{-3} + (2x-1)^2(-3)(3x+2)^{-4}(3)$
$= 4(2x-1)(3x+2)^{-3} - 9(3x+2)^{-4}(2x-1)^2$
$= (2x-1)(3x+2)^{-4}[4(3x+2)-9(2x-1)]$
$= (2x-1)(3x+2)^{-4}(-6x+17)$
$= \dfrac{(2x-1)(-6x+17)}{(3x+2)^4}$

Quotient Rule

If $h(x)=\dfrac{f(x)}{g(x)}$, then

$$h'(x)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}, g(x)\neq 0$$

In Leibniz notation, $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{\dfrac{du}{dx}v-u\dfrac{dv}{dx}}{v^2}$

Proof:

Since $h(x)=\dfrac{f(x)}{g(x)}, g(x)\neq 0$, therefore

$h(x)g(x)=f(x)$ → multiply each side by $g(x)$

$h'(x)g(x)+h(x)g'(x)=f'(x)$ →differentiate each side

$$h'(x)g(x)=f'(x)-h(x)g'(x)$$

$$h'(x)=\frac{f'(x)-h(x)g'(x)}{g(x)}$$

$$=\frac{f'(x)-\dfrac{f(x)}{g(x)}g'(x)}{g(x)}$$

$$=\frac{f'(x)-\dfrac{f(x)}{g(x)}g'(x)}{g(x)}\times\frac{g(x)}{g(x)}$$

$$=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

Examples

1. Determine the derivative of $h(x)=\dfrac{3x-4}{x^2+5}$

$$h'(x)=\frac{3(x^2+5)-(3x-4)(2x)}{(x^2+5)^2}$$

$$=\frac{3x^2+15-6x^2+8x}{(x^2+5)^2}$$

$$=\frac{-3x^2+8x+15}{(x^2+5)^2}$$

2. Determine the equation of the **normal** to $y=\dfrac{2x}{x^2+1}$ at $x=0$.

Definition: A normal line to the graph of a function f(x) is defined to be the line perpendicular to the tangent at a given point

$$y'=f'(x)=\frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$$

$$=\frac{-2x^2+2}{(x^2+1)^2}$$

$$f'(0)=\frac{-2(0)^2+2}{[(0)^2+1]^2}$$

$$=2$$

$\therefore m_{\perp}=-\frac{1}{2}$

$$f(0)=\frac{2(0)}{(0)^2+1}$$

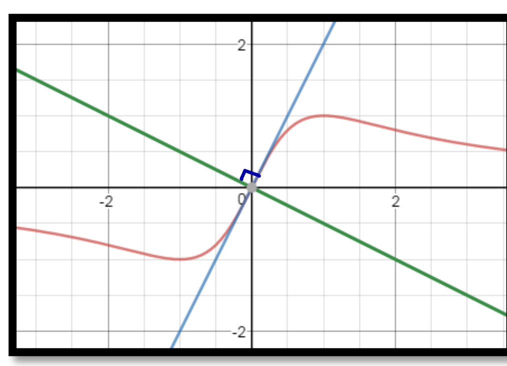
$$=0$$

Equation of normal is

$$y-y_1=m(x-x_1)$$

$$y-0=-\frac{1}{2}(x-0)$$

$$\boxed{y=-\frac{1}{2}x}$$



3. Determine the coordinates of each point on the graph of $f(x)=\dfrac{2x+8}{\sqrt{x}}$ where the tangent is horizontal.

$$f'(x)=\frac{2(x^{\frac{1}{2}})-(2x+8)(\frac{1}{2}x^{-\frac{1}{2}})}{x}$$
$$=\frac{2x^{\frac{1}{2}}-x^{-\frac{1}{2}}(x+4)}{x}$$
$$=\frac{2x^{\frac{1}{2}}-x^{\frac{1}{2}}-4x^{-\frac{1}{2}}}{x}$$
$$=\frac{x^{\frac{1}{2}}-4x^{-\frac{1}{2}}}{x}$$
$$=\frac{x^{-\frac{1}{2}}(x-4)}{x}$$
$$=\frac{x-4}{x^{\frac{3}{2}}}$$

OR using product rule:

$$f(x)=2x^{\frac{1}{2}}+8x^{-\frac{1}{2}}$$
$$f'(x)=x^{-\frac{1}{2}}-4x^{-\frac{3}{2}}$$
$$=x^{-\frac{3}{2}}(x-4)$$
$$=\frac{x-4}{x^{\frac{3}{2}}}$$

$\because$ Tangent is horizontal $\Rightarrow m_t=0$

$0=x-4$

$\boxed{x=4}$

$$f(4)=\frac{2(4)+8}{\sqrt{4}}$$
$$=8$$

$\therefore$ The point is $(4,8)$

Challenging Questions

1. Determine the equation of the tangent line to $g(x)=\left(\dfrac{1}{x^3}+1\right)(x-1)$ at $x=-1$.

2. If $g(x)=\dfrac{f(x)}{\sqrt{x-1}}$, where $f(5)=8$, and $f'(5)=-5$, find $g'(5)$.

3. Find the points on the function $f(x)=\dfrac{x+9}{x+8}$ where the tangent lines pass through the origin

4. Recall: **A normal line to the graph of a function f(x) is defined to be the line perpendicular to the tangent at a given point.** Find the equation of the normal to the curve $y=\sqrt[3]{x^2-1}$ at the point where x=3 .

5. Let f and g be functions such that $g(x)=\dfrac{f(x)}{x}$. If $y=2x-3$ is the equation of the tangent to the graph of f(x) at x=1, what is the equation of the line tangent to the graph of g(x) at x=1 ?

6. Find the points on the curve $y=\dfrac{x}{x+1}$ where the **normal** line is parallel to $x+y=2$.

Differentiate the following. Write the final answer with positive exponents.

$$h(x)=\frac{(2x-1)^2}{(3x+2)^3}$$

$$\mathbf{h(x)=(2x-1)^2(3x+2)^{-3}}$$

$$\mathbf{h'(x)=2(2x-1)(2)(3x+2)^{-3}+(-3)(3x+2)^{-4}(3)(2x-1)^2}$$

$$\mathbf{=4(2x-1)(3x+2)^{-3}-9(3x+2)^{-4}(2x-1)^2}$$

$$\mathbf{=(2x-1)(3x+2)^{-4}\left[4(3x+2)-9(2x-1)\right]}$$

$$\mathbf{=(2x-1)(3x+2)^{-4}\left[12x+8-18x+9\right]}$$

$$\mathbf{=(2x-1)(3x+2)^{-4}(17-6x)}$$

$$\mathbf{=\frac{(2x-1)(17-6x)}{(3x+2)^4}}$$

Challenging Questions-Solutions

1. Determine the equation of the tangent line to $g(x)=\left(\frac{1}{x^3}+1\right)(x-1)$ at $x=-1$.

$x=-1$, $g(-1)=0$

$g(x)=\left(x^{-3}+1\right)(x-1)$

$g'(x)=\left(-3x^{-4}\right)(x-1)+\left(x^{-3}+1\right)$

$m_T=g'(-1)=(-3)(-2)+(0)$

$m_T=6$

equation of tangent line:

$y-0=6(x+1)$

$\therefore y=6x+6$

2. If $g(x)=\dfrac{f(x)}{\sqrt{x-1}}$, where $f(5)=8$, and $f'(5)=-5$, find $g'(5)$.

$g(x)=f(x)(x-1)^{\frac{-1}{2}}$

$g'(x)=f'(x)(x-1)^{\frac{-1}{2}}+\left(\frac{-1}{2}\right)(x-1)^{\frac{-3}{2}}f(x)$

$g'(5)=f'(5)(5-1)^{\frac{-1}{2}}+\left(\frac{-1}{2}\right)(5-1)^{\frac{-3}{2}}f(5)$

$=(-5)\left(\frac{1}{2}\right)+\left(\frac{-1}{2}\right)\left(\frac{1}{8}\right)(8)$

$=-3$

3. Find the points on the function $f(x)=\dfrac{x+9}{x+8}$ where the tangent lines pass through the origin.

$f(x)=(x+9)(x+8)^{-1}$

$f'(x)=(x+8)^{-1}+(-1)(x+8)^{-2}(x+9)$

$f'(x)=(x+8)^{-2}\left[(x+8)-(x+9)\right]$

$m_T=f'(x)=\dfrac{-1}{(x+8)^2}$

Slope of tangent line passes through origin and point $\left(x,\dfrac{x+9}{x+8}\right)$ is

$m=\dfrac{y_2-y_1}{x_2-x_1}$

$m_T=\dfrac{\frac{x+9}{x+8}-0}{x-0}\rightarrow m_T=\dfrac{x+9}{x(x+8)}$

$\because m_t=f'(x)$

$\therefore \dfrac{x+9}{x(\cancel{x+8})}=\dfrac{-1}{(x+8)^{\cancel{2}}}$

$(x+8)(x+9)=-x$

$x^2+18x+72=0$

$(x+6)(x+12)=0$

$x=-6 \quad or \quad x=-12$

$\therefore$ points are $\left(-6,\dfrac{3}{2}\right)$ and $\left(-12,\dfrac{3}{4}\right)$

4. **A normal line to the graph of a function f(x) is defined to be the line perpendicular to the tangent at a given point.** Find the equation of the normal to the curve $y=\sqrt[3]{x^2-1}$ at the point where x=3 .

$y=(x^2-1)^{\frac{1}{3}}$

$y'=\dfrac{1}{3}(x^2-1)^{-\frac{2}{3}}(2x)$

At x=3: $m_T=\dfrac{1}{3}(3^2-1)^{-\frac{2}{3}}(6)$

$m_T=\dfrac{1}{2}\rightarrow m_\perp=-2$

At $x=3$, $y=\sqrt[3]{(3)^2-1}$
$=2$

$y=-2x+b$

$2=-6+b$

$b=8$

Equation of normal line $\therefore y=-2x+8$

5. Let f and g be functions such that $g(x)=\dfrac{f(x)}{x}$. If y = 2x – 3 is the equation of the tangent to the graph of f(x) at x=1, what is the equation of the line tangent to the graph of g(x) at x=1 ?

$g(x)=x^{-1}f(x)$

$g'(x)=-x^{-2}f(x)+f'(x)x^{-1}$

$m_T=g'(1)=-f(1)+f'(1)$

At $x=1$, $m=2$ that means $f'(1)=2$.

Also at $x=1$, $y=2(1)-3=-1$

that means $f(1)=-1$

$m_T=g'(1)=-f(1)+f'(1)$
$=-(-1)+2$
$=3$

At $x=1$, $g(1)=\dfrac{f(1)}{1}=-1$.

Equation of tangent line at $(1,-1)$ on $g(x)$ is

$y=3x+b$

$-1=3+b$

$b=-4$

$\therefore y=3x-4$

6. Find the points on the curve $y=\dfrac{x}{x+1}$ where the normal line is parallel to $x+y=2$.

$x+y=2 \rightarrow y=-x+2 \quad m_{\perp}=-1 \Rightarrow m_t=1$

$$y'=\frac{1}{(x+1)^2}$$

$$\frac{1}{(x+1)^2}=1$$

$$(x+1)^2=1$$

$$x+1=\pm 1$$

$$x=-1\pm 1$$

$\therefore x=0$ or $x=-2$

At $x=0$, $y=\dfrac{(0)}{(0)+1}$
$=0$

At $x=-2$, $y=\dfrac{(-2)}{(-2)+1}$
$=2$

$\therefore$ The points are $(0,0)$ and $(-2,2)$.

Warm-Up: PRODUCT RULE

The limit below represents the derivative of some function f(x) evaluated at some number $\boldsymbol{a}$. Determine the function and the number $\boldsymbol{a}$.

$$f'(a)=\lim_{h\to 0}\frac{2(6+h)^2-2(6)^2}{h}\quad,\quad f(x)=\underline{2x^2}\quad,\quad \boldsymbol{a}=\underline{6}$$

1. Differentiate the following .Where applicable; write the final answers with positive exponents.

a) $g(x)=\left(5\sqrt[5]{x^3}-\dfrac{1}{2x^2}\right)\sqrt[3]{x}$

$g(x)=\left(5x^{\frac{3}{5}}-\frac{1}{2}x^{-2}\right)x^{\frac{1}{3}}$

$g(x)=5x^{\frac{14}{15}}-\frac{1}{2}x^{-\frac{5}{3}}$

$g'(x)=\frac{14}{3}x^{-\frac{1}{15}}+\frac{5}{6}x^{-\frac{8}{3}}$

$g'(x)=\dfrac{14}{3x^{\frac{1}{15}}}+\dfrac{5}{6x^{\frac{8}{3}}}$

b) $g(t)=\dfrac{\pi t^5-2t^{-4}+3\pi^2}{3t^2}$

$g(t)=\frac{\pi}{3}t^3-\frac{2}{3}t^{-6}+\pi^2t^{-2}$

$g'(t)=\pi t^2+\dfrac{4}{t^7}-\dfrac{2\pi^2}{t^3}$

2. Given $f'(1)=4, g'(1)=-2, f(1)=1$, and $g(1)=1$, find $h'(1)$ if $h(x)=\underline{\left(2x-\sqrt{x}\right)^2g(x)}+\underline{x^3f(x)}$.

$h'(x)=\underline{2\left(2x-\sqrt{x}\right)\left(2-\frac{1}{2\sqrt{x}}\right)g(x)+\left(2x-\sqrt{x}\right)^2g'(x)}+\underline{3x^2f(x)+x^3f'(x)}$

$h'(1)=2\left[2(1)-\sqrt{(1)}\right]\left[2-\frac{1}{2\sqrt{(1)}}\right]g(1)+\left[2(1)-\sqrt{(1)}\right]^2g'(1)+3(1)^2f(1)+(1)^3f'(1)$

$=2(1)\left(\frac{3}{2}\right)(1)+(-2)(1)+3(1)+1(4)$

$=3-2+3+4$

$=8$

The Chain Rule

If $g(x)$ is differentiable at x and $f(x)$ is differentiable at $g(x)$, then the composite function, $h(x)=f(g(x))$ or $h(x)=(fog)(x)$ is differentiable at x and $h'(x)$ is given by the product

$$h'(x)=f'(g(x))g'(x).$$

In Leibniz notation, If y=f(u) and u=g(x), are both differentiable functions, then

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Proof of Chain Rule:

$$[f(g(x))]' = \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}$$

$$=\lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{h}\times 1\right]$$

$$=\lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{h}\times\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right]$$

We can only make this move if we know that $g(x+h)-g(x)\neq 0$. In other words, this proof is not valid over any domain of the function for which the graph of $y=g(x)$ is a straight horizontal line.

$$=\lim_{h\to 0}\left[\left(\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right)\left(\frac{g(x+h)-g(x)}{h}\right)\right]$$

$$=\lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

Look at the denominator of the first fraction. We're taking the limit of that fraction as $h\to 0$. We know that $\lim_{h\to 0}[g(x+h)-g(x)]=0$. So, we'll let $g(x+h)-g(x)=k$. Recognizing that $k\to 0$ as $h\to 0$, we're able to rewrite that last line as follows:

$$=\lim_{k\to 0}\left[\frac{f(g(x)+k)-f(g(x))}{k}\right]\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$=f'(g(x))g'(x)$$

Examples:

1. If $y = u^2 + u - 1$, and if $u = x^2 - 2\sqrt{x}$, then evaluate $\frac{dy}{dx}$ at $x = 1$

$\frac{dy}{du} = 2u + 1$

$\frac{du}{dx} = 2x - x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = (2u+1)\left(2x - \frac{1}{\sqrt{x}}\right)$

At $x=1$, $u = 1-2$

$\boxed{u=-1}$

$\left.\frac{dy}{dx}\right|_{x=1, u=1} = [2(-1)+1]\left[2(1) - \frac{1}{\sqrt{(1)}}\right]$

$= (-1)(1)$

$= -1$

2. Determine the derivative of $f(x) = \sqrt[3]{\frac{x^2-3}{3-5x}}$

$f(x) = \left(\frac{x^2-3}{3-5x}\right)^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}\left(\frac{x^2-3}{3-5x}\right)^{-\frac{2}{3}}\left[\frac{2x(3-5x) - (x^2-3)(-5)}{(3-5x)^2}\right]$

$= \frac{1}{3}\left(\frac{x^2-3}{3-5x}\right)^{-\frac{2}{3}}\left[\frac{6x - 10x^2 + 5x^2 - 15}{(3-5x)^2}\right]$

$= \frac{1}{3} \cdot \frac{(3-5x)^{\frac{2}{3}}}{(x^2-3)^{\frac{2}{3}}} \cdot \frac{(-5x^2+6x-15)}{(3-5x)^2}$

$= \frac{-5x^2+6x-15}{3(x^2-3)^{\frac{2}{3}}(3-5x)^{\frac{4}{3}}}$

3. Differentiate:

a) $f(x) = m\left(nx^2 + rx\right)^{\sqrt{7}}$

$f'(x) = \sqrt{7}\,m(nx^2+rx)^{\sqrt{7}-1}(2nx+r)$

b) $f(x) = (5+3x)^{\pi}$

$f'(x) = \pi(5+3x)^{\pi-1}(3)$

$= 3\pi(5+3x)^{\pi-1}$

c) $y = \frac{x}{\sqrt{1-4x^2}}$

$y' = \frac{1(1-4x^2)^{\frac{1}{2}} - x\left(\frac{1}{2}\right)(1-4x^2)^{-\frac{1}{2}}(-8x)}{1-4x^2}$

$y' = \frac{(1-4x^2)^{\frac{1}{2}} + 4x^2(1-4x^2)^{-\frac{1}{2}}}{1-4x^2}$

$y' = \frac{(1-4x^2)^{-\frac{1}{2}}\left[(1-4x^2)+4x^2\right]}{1-4x^2}$

$y' = \frac{1}{(1-4x^2)^{\frac{3}{2}}}$

d) $f(x) = \sqrt{4x^2 + 6x - 1}$

$f'(x) = \frac{1}{2}(4x^2+6x-1)^{-\frac{1}{2}}(8x+6)$

$= \frac{4x+3}{\sqrt{4x^2+6x-1}}$

e) $y=(2x^2-9)\sqrt{3x^2+5x}$

$$y=(2x^2-9)(3x^2+5x)^{\frac{1}{2}}$$

$$y'=4x(3x^2+5x)^{\frac{1}{2}}+(2x^2-9)\left(\tfrac{1}{2}\right)(3x^2+5x)^{-\frac{1}{2}}(6x+5)$$

$$y'=\tfrac{1}{2}(3x^2+5x)^{-\frac{1}{2}}\left[8x(3x^2+5x)+(6x+5)(2x^2-9)\right]$$

$$y'=\frac{24x^3+40x^2+12x^3-54x+10x^2-45}{2(3x^2+5x)^{\frac{1}{2}}}$$

$$y'=\frac{36x^3+50x^2-54x-45}{2\sqrt{3x^2+5x}}$$

f) $m(t)=\sqrt[3]{t+\sqrt{1+t^2}}$

$$m(t)=\left[t+(1+t^2)^{\frac{1}{2}}\right]^{\frac{1}{3}}$$

$$m'(t)=\tfrac{1}{3}\left(t+\sqrt{1+t^2}\right)^{-\frac{2}{3}}\left[1+\tfrac{1}{2}(1+t^2)^{-\frac{1}{2}}(2t)\right]$$

$$=\frac{1+\frac{t}{\sqrt{1+t^2}}}{3\left(t+\sqrt{1+t^2}\right)^{\frac{2}{3}}}$$

$$=\frac{\sqrt{1+t^2}+t}{3\left(t+\sqrt{1+t^2}\right)^{\frac{2}{3}}\left(\sqrt{1+t^2}\right)}$$

4. a) If $h(x)=\dfrac{(f(x))^2}{g(x)}$, determine $h'(x)$.

$$h'(x)=\frac{2f(x)\cdot f'(x)\,g(x)-(f(x))^2g'(x)}{[g(x)]^2}$$

$$=\frac{f(x)\left[2f'(x)\cdot g(x)-f(x)\,g'(x)\right]}{[g(x)]^2}$$

b) Given $f(1)=2$, $f'(1)=-3$, $g(1)=1$ and $g'(1)=4$ find $h'(1)$.

$$h'(1)=\frac{2f(1)\,f'(1)\,g(1)-(f(1))^2\cdot g'(1)}{[g(1)]^2}$$

$$h'(1)=\frac{2(2)(-3)(1)-(4)(2)^2}{1^2}$$

$$h'(1)=-12-16$$

$$\boxed{h'(1)=-28}$$

5. If $y=f(3x^4)$ and $f'(3)=\dfrac{-1}{4}$, determine $\left.\dfrac{dy}{dx}\right|_{x=1}$.

$$\frac{dy}{dx}=f'(3x^4)\cdot(12x^3)$$

$$\left.\frac{dy}{dx}\right|_{x=1}=f'(3)\cdot(12)$$

$$=\left(-\tfrac{1}{4}\right)(12)$$

$$=-3$$

Practice

1. Given that $g(2)=4, g'(2)=-1, h(2)=2,$ and $h'(2)=3,$ find $f'(2)$ if

a) $f(x)=(g(x))^3$

$f'(x)=3[g(x)]^2 \cdot g'(x)$

$f'(2)=3[g(2)]^2 \cdot g'(2)$

$=3(4)^2 \cdot (-1)$

$=-48$

b) $f(x)=g(h(x))$

$f'(x)=g'[h(x)] \cdot h'(x)$

$f'(2)=g'[h(2)] \cdot h'(2)$

$=g'(2)\cdot(3)$

$=(-1)(3)$

$=-3$

2. Let $f(x)=h(g(x))$ and $j(x)=h(x)g(x),$ where g and h are differentiable functions on $\mathbb{R}$. Fill in the missing entries on the table below and determine $h(4)$ and $h'(4)$.

x	h(x)	h′(x)	g(x)	g′(x)	f(x)	f′(x)	j(x)	j′(x)
0	−4	**1**	**2**	−2	1	**−8**	**−8**	**10**
1	**2**	2	2	1	1	**4**	**4**	**6**
2	1	**4**	**4**	2	**15**	**8**	**4**	**18**

$h(0)=\frac{j(0)}{g(0)}$
$=\frac{-8}{2}$
$=-4$

$h(2)=\frac{j(2)}{g(2)}$
$=\frac{4}{4}$
$=1$

$g(1)=\frac{j(1)}{h(1)}$
$=\frac{4}{2}$
$=2$

$f(0)=h(g(0))$
$=h(2)$
$=1$

$f(1)=h(g(1))$
$=h(2)$
$=1$

$f'(x)=h'(g(x))\cdot g'(x)$

$j'(x)=h'(x)g(x)+h(x)g'(x)$

$f'(0)=h'(g(0))\cdot g'(0)$
$(-8)=h'(2)\cdot g'(0)$
$-8=(4)\cdot g'(0)$
$-2=g'(0)$

$j'(0)=h'(0)g(0)+h(0)g'(0)$
$(10)=(1)(2)+(-4)g'(0)$
$\frac{10-2}{-4}=g'(0)$
$-2=g'(0)$

$f'(1)=h'(g(1))\cdot g'(1)$
$(-4)=h'(2)\cdot g'(1)$
$-4=(4)\cdot g'(1)$
$-1=g'(1)$

$j'(1)=h'(1)g(1)+h(1)g'(1)$
$(6)=h'(1)(2)+(2)(-1)$
$\frac{6+2}{2}=h'(1)$
$4=h'(1)$

$j'(2)=h'(2)g(2)+h(2)g'(2)$
$(18)=(4)(4)+(1)g'(2)$
$18-16=g'(2)$
$2=g'(2)$

$f(2)=h(g(2))$
$15=h(4)$
$\therefore h(4)=15$

$f'(2)=h'(g(2))\cdot g'(2)$
$(8)=h'(4)\cdot g'(2)$
$8=h'(4)\cdot(2)$
$4=h'(4)$

SOLUTIONS

3. Slope of the normal to the curve with equation $y = ax + \frac{b}{4-3x}$ at point $(1,6)$ is $-\frac{1}{2}$. Find the values of *a* and *b*.

$m_\perp = -\frac{1}{2}$

$\therefore m_t = 2$

$y = ax + b(4-3x)^{-1}$

$y' = x + b(-1)(4-3x)^{-2}(-3)$

$= x + \frac{3b}{(4-3x)^2}$

$\because m_t = 2$ at $x = 1$,

$2 = a + \frac{3b}{[4-3(1)]^2}$

$2 = a + 3b \rightarrow ①$

At $(1,6)$: $(6) = a(1) + \frac{b}{4-3(1)}$

$6 = a + b \rightarrow ②$

② − ①: $4 = -2b$

$-2 = b$

sub $b = -2$ into ①

$2 = a + 3(-2)$

$8 = a$

$\therefore a = 8,\ b = -2$

4. Find $\boldsymbol{k}$ given that the tangent to $f(x) = \frac{4}{(\boldsymbol{k}x+1)^2}$ at $x = 0$ passes through (1, 0)

$f(0) = \frac{4}{[k(0)+1]^2}$

$= 4$ $\quad\therefore$ Pt of tangency $(0,4)$

$f(x) = 4(kx+1)^{-2}$

$\therefore f'(x) = -8(kx+1)^{-3}(k)$

$= -8k(kx+1)^{-3}$

$f'(0) = -8k[k(0)+1]^{-3}$

$= -8k$

$\because$ Tangent passes through $(0,4)$ and $m_t = -8k$

$\therefore$ Eq'n of tangent: $y = -8kx + 4$

$\because$ Tangent line passes through $(1,0)$, sub in to find k:

$(0) = -8k(1) + 4$

$-4 = -8k$

$\frac{1}{2} = k$

SOLUTIONS

5. Consider $f(x)=\dfrac{4}{\sqrt{4-x}}$.

(a) Find the equations of the tangent and normal at the point where P(3,4).

(b) If the tangent line cuts the x-axis at A and the normal line cuts the x-axis at B, find the coordinates of A and B.

(c) Find the area of triangle PAB.

a) $f(x) = 4(4-x)^{-\frac{1}{2}}$

$\therefore f'(x) = -2(4-x)^{-\frac{3}{2}}(-1)$

$= 2(4-x)^{-\frac{3}{2}}$

$m_t = f'(3) = 2[4-(3)]^{-\frac{3}{2}}$

$= 2$

$\therefore$ Equ'n of tangent:

$y-(4) = (2)[x-(3)]$

$y = 2x-2$

Normal:

$m_\perp = -\frac{1}{2}$, point $(3,4)$

$y-(4) = (-\frac{1}{2})[x-(3)]$

$y = -\frac{1}{2}x + \frac{11}{2}$

b) Find x-intercepts for the tangent and normal line:

Tangent:

$0 = 2x - 2$

$1 = x$

$\therefore$ Point A is $(1,0)$

Normal:

$0 = -\frac{1}{2}x + \frac{11}{2}$

$x = 11$

$\therefore$ Point B is $(11,0)$.

c) Base is $|x_B - x_A|$

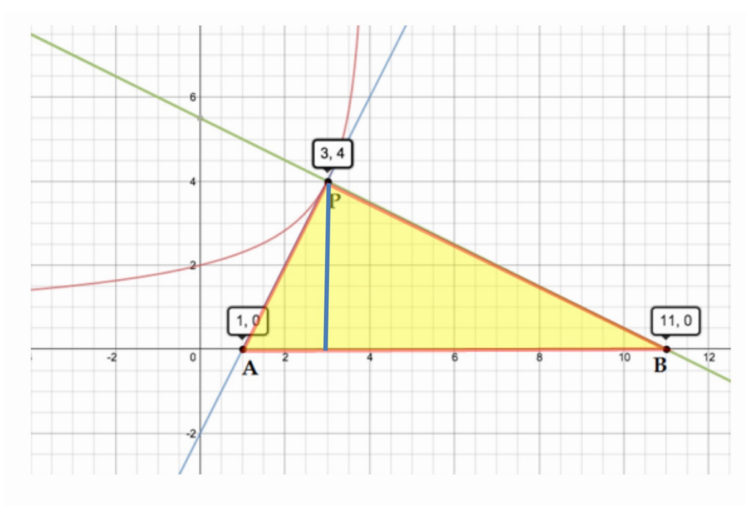
$= |11-1|$

$= 10$ units

Height is the y-coordinate of $P = 4$

$\therefore A = \frac{1}{2}(10)(4)$

$= 20$ units^2



More Practice on using Chain Rule

1. Differentiate the following .Where applicable; write the final answers with positive exponents.

a) $y=\left(2+x^{2}\right)^{\pi}+\sqrt[5]{1-x^{2}}+\dfrac{3x^{2}-\sqrt{x}+5}{\sqrt{5x}}$

b) $S(t)=t^{2}\left(1-\dfrac{2\pi}{t^{2}}\right)+\sqrt{4t^{2}-5}$

c) $h(x)=\dfrac{(3x+2)^{-3}}{(2x-1)^{-2}}$ ***(Express in a fully factored form)***

2. Let f and g be differentiable functions such that $g(1)=3, f(1)=-2$, $f(3)=-1$, $f'(3)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{f(g(x))}{f(x)+g(x)}$. If $h'(1)=5$, find the value of $f'(1)$.

3. Find the points on the curve $y=\left(1-\dfrac{x}{5}\right)^{3}$ where the slope of normal line is 15.

4. If $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=-4$ and $g(x)=f\left(\sqrt{5-x^{2}}\right)$, determine the value of $g'(1)$.

5 . Assume that $h(x)=\left[f(x)\right]^{3}.g(x)$, where f and g are differentiable functions

If $f(0)=\dfrac{-1}{2}, f'(0)=\dfrac{-8}{3}$ and $g(0)=-1, g'(0)=-2$, determine an equation of the line tangent to the graph of h at x=0 .

6. Consider the curve $\mathbf{y=a\sqrt{x}+\dfrac{b}{\sqrt{x}}}$ where ***a*** and ***b*** are constants. The normal to this curve at the point where x = 4 is 4x + *y* = 22. Find the values of *a* and *b*.

7. Line $y=k$ is tangent to the curve $f(x)=\dfrac{x^{2}-5}{x-(k+1)}$ at x=1. Find value of k.

8. Given $y=\dfrac{u+3}{2u-1}$, and $u=\sqrt{x^{2}+3}$, determine $\left.\dfrac{dy}{dx}\right|_{x=1}$ by using the Leibniz notation.

More Practice on using Chain Rule

1. Differentiate the following .Where applicable; write the final answers with positive exponents.

a) $y=\left(2+x^2\right)^{\pi}+\sqrt[5]{1-x^2}+\dfrac{3x^2-\sqrt{x}+5}{\sqrt{5x}}$ ← Divide

$$y=(2+x^2)^{\pi}+(1-x^2)^{\frac{1}{5}}+\frac{3\sqrt{5}}{5}x^{\frac{3}{2}}-\frac{\sqrt{5}}{5}+\sqrt{5}x^{-\frac{1}{2}}$$

$$y'=\pi(2+x^2)^{\pi-1}(2x)+\frac{1}{5}(1-x^2)^{-\frac{4}{5}}(-2x)+\frac{3\sqrt{5}}{5}\left(\frac{3}{2}\right)x^{\frac{1}{2}}+\sqrt{5}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$=2\pi x(2+x^2)^{\pi-1}-\frac{2x}{5(1-x^2)^{\frac{4}{5}}}+\frac{9\sqrt{5}}{10}x^{\frac{1}{2}}-\frac{\sqrt{5}}{2x^{\frac{3}{2}}}$$

b) $S(t)=t^2\left(1-\dfrac{2\pi}{t^2}\right)+\sqrt{4t^2-5}$

$$S(t)=t^2-2\pi+(4t^2-5)^{\frac{1}{2}}$$

$$S'(t)=2t+\frac{1}{2}(4t^2-5)^{-\frac{1}{2}}(8t)$$

$$=2t+\frac{4t}{\sqrt{4t^2-5}}$$

c) $h(x)=\dfrac{(3x+2)^{-3}}{(2x-1)^{-2}}$ ***(Express in a fully factored form)***

$$h(x)=\frac{(2x-1)^2}{(3x+2)^3}$$

$$h'(x)=\frac{2(2x-1)(2)(3x+2)^3-(2x-1)^2(3)(3x+2)^2(3)}{(3x+2)^6}$$

$$=\frac{4(2x-1)(3x+2)^3-9(2x-1)^2(3x+2)^2}{(3x+2)^6}$$

factor

$$=\frac{(2x-1)(3x+2)^2\left[4(3x+2)-9(2x-1)\right]}{(3x+2)^6}$$

simplified $(3x+2)^2$

$$=\frac{(2x-1)(12x+8-18x+9)}{(3x+2)^4}$$

$$=\frac{(2x-1)(-6x+17)}{(3x+2)^4}$$

2. Let f and g be differentiable functions such that $g(1)=3$, $f(1)=-2$, $f(3)=-1$, $f'(3)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{f(g(x))}{f(x)+g(x)}$. If $h'(1)=5$, find the value of $f'(1)$.

$$h'(x)=\frac{f'(g(x))g'(x)(f(x)+g(x))-(f'(x)+g'(x))f(g(x))}{(f(x)+g(x))^2}$$

$$h'(1)=\frac{f'(g(1))g'(1)(f(1)+g(1))-(f'(1)+g'(1))f(g(1))}{(f(1)+g(1))^2}$$

$$5=\frac{(2)(f'(3))(1)-(2+f'(1))f(3)}{1^2} \rightarrow (5)=\frac{f'(3)\cdot(2)[(-2)+(3)]-[f'(1)+(2)]\cdot f(3)}{[(-2)+(3)]^2}$$

$$5=(-2)(2)-[f'(1)+2](-1)$$

$$5=-4+f'(1)+2$$

$$7=f'(1)$$

$$5=-4+2+f'(1)$$

$$\boxed{f'(1)=7}$$

3. Find the points on the curve $y=\left(1-\dfrac{x}{5}\right)^3$ where the slope of normal line is 15.

$$m_\perp=15 \Rightarrow m_t=-\frac{1}{15}$$

$$y'=3\left(1-\frac{x}{5}\right)^2\left(\frac{-1}{5}\right)$$

$$-\frac{1}{15}=-\frac{3}{15}\left(1-\frac{x}{5}\right)^2$$

$$\frac{1}{9}=\left(1-\frac{x}{5}\right)^2$$

$$\pm\frac{1}{3}=1-\frac{x}{5}$$

$$\frac{1}{3}=1-\frac{x}{5} \quad \text{or} \quad -\frac{1}{3}=1-\frac{x}{5}$$

$$\frac{x}{5}=1-\frac{1}{3} \qquad \frac{x}{5}=1+\frac{1}{3}$$

$$\frac{x}{5}=\frac{2}{3} \qquad \frac{x}{5}=\frac{4}{3}$$

$$\boxed{x=\frac{10}{3}}\rightarrow\boxed{y=\frac{1}{27}} \qquad \boxed{x=\frac{20}{3}}\rightarrow\boxed{y=-\frac{1}{27}}$$

4. If $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=-4$ and $g(x)=f\left(\sqrt{5-x^2}\right)$, determine the value of $g'(1)$.

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=-4 \rightarrow f'(2)=-4$$

$$g(x)=f\left(\sqrt{5-x^2}\right)$$

$$g'(x)=f'\left(\sqrt{5-x^2}\right)\bullet\frac{-2x}{2\sqrt{5-x^2}}$$

$$g'(1)=f'\left(\sqrt{5-(1)^2}\right)\cdot\frac{-2(1)}{2\sqrt{5-(1)^2}}$$

$$g'(1)=f'(2)\bullet\frac{-1}{2}$$

$$=(-4)\left(\frac{-1}{2}\right)$$

$$=2$$

5 . Assume that $h(x)=[f(x)]^3.g(x)$, where f and g are differentiable functions

If $f(0)=\frac{-1}{2}, f'(0)=\frac{-8}{3}$ and $g(0)=-1, g'(0)=-2$, determine an equation of the line tangent to the graph of h at x=0 .

$$h'(x)=3[f(x)]^2.f'(x).g(x)+g'(x).[f(x)]^3$$

$$h'(o)=3[f(o)]^2.f'(o).g(o)+g'(o).[f(o)]^3$$

$$=3\left(\frac{-1}{2}\right)^2\left(\frac{-8}{3}\right)(-1)+(-2)\left(\frac{-1}{2}\right)^3$$

$$=2+\frac{1}{4}$$

$$=\frac{9}{4} \leftarrow m_t$$

$$h(o)=[f(o)]^3 g(o)$$

$$=\left(\frac{-1}{2}\right)^3(-1)$$

$$=\frac{1}{8} \rightarrow (0,\tfrac{1}{8}) \text{ is point of tangency}$$

Equation: $y-\frac{1}{8}=\frac{9}{4}x \rightarrow \boxed{y=\frac{9}{4}x+\frac{1}{8}}$

6. Consider the curve $y=a\sqrt{x}+\frac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where x = 4 is 4x + y = 22. Find the values of a and b.

$$y=ax^{\frac{1}{2}}+bx^{\frac{-1}{2}}$$

$$y=x^{\frac{-1}{2}}(ax+b)$$

$$y'=\frac{-1}{2}x^{\frac{-3}{2}}(ax+b)+ax^{\frac{-1}{2}}$$

At $x=4$: $m_t=\frac{-1}{2}(4)^{\frac{-3}{2}}(4a+b)+a(4)^{\frac{-1}{2}}$

$$m_t=\frac{-1}{16}(4a+b)+\frac{1}{2}a$$

$$m_t=\frac{1}{4}a-\frac{1}{16}b$$

$\because$ $4x+y=22 \rightarrow y=-4x+22$

$$m_\perp=-4 \rightarrow m_t=\frac{1}{4}$$

$$\frac{1}{4}a-\frac{1}{16}b=\frac{1}{4} \rightarrow 4a-b=4 \quad (1)$$

$$y=a\sqrt{x}+\frac{b}{\sqrt{x}} \xrightarrow{\text{at } x=4} y=2a+\frac{b}{2}$$

$$4x+y=22 \xrightarrow{\text{at } x=4} y=6$$

Point on the curve where the normal occurs

$$2a+\frac{b}{2}=6$$

$$4a+b=12 \quad (2)$$

$$\left.\begin{array}{l} 4a-b=4 \quad (1) \\ 4a+b=12 \quad (2) \end{array}\right\rangle a=2\ ,\ b=4$$

horizontal line $\therefore m_t = 0$

7. Line $y = k$ is tangent to the curve $f(x) = \dfrac{x^2 - 5}{x - (k+1)}$ at x=1. Find value of k.

$$f'(x) = \frac{2x(x-k-1) - x^2 + 5}{(x-k-1)^2}$$

$$f'(1) = 0 \rightarrow 2(1-k-1) - 1 + 5 = 0$$

$$-2k = -4$$

$$\boxed{k = 2}$$

8. Given $y = \dfrac{u+3}{2u-1}$, and $u = \sqrt{x^2+3}$, determine $\left.\dfrac{dy}{dx}\right|_{x=1}$ by using the Leibniz notation.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\left(\frac{-7}{(2u-1)^2}\right) \times \left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}}\right)\right|_{x=1, u=2}$$

$$= \frac{-7}{9} \times \frac{1}{2}$$

$$= -\frac{7}{18}$$

$$y = (u+3)(2u-1)^{-1}$$

$$\frac{dy}{du} = (2u-1)^{-1} + (u+3)(-1)(2u-1)^{-2}(2)$$

$$= \frac{1}{2u-1} - \frac{2(u+3)}{(2u-1)^2}$$

$$= \frac{2u-1-2u-6}{(2u-1)^2}$$

$$= \frac{-7}{(2u-1)^2}$$

$$u = (x^2+3)^{\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2}(x^2+3)^{-\frac{1}{2}}(2x)$$

$$= \frac{x}{(x^2+3)^{\frac{1}{2}}}$$

At $x = 1$, $u = [(1)^2 + 3]^{\frac{1}{2}}$

$$= 2$$

Mid-Review

1. Let f and g be differentiable functions such that $g(1)=3, f(1)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{(fg)(x)}{f(x)+g(x)}$. If $h'(1)=5$, find the value of $f'(1)$.

2. Find the points on the curve $y=\left(1-\dfrac{x}{5}\right)^3$ where the slope of normal line is 15.

3. If $\displaystyle\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=-4$ and $g(x)=f\left(\sqrt{5-x^2}\right)$, determine the value of $g'(1)$.

4. Assume that $h(x)=\left[f(x)\right]^3.g(x)$, where f and g are differentiable functions.
If $f(0)=\dfrac{-1}{2}, f'(0)=\dfrac{-8}{3}$ and $g(0)=-1, g'(0)=-2$, determine an equation of the line tangent to the graph of h at x=0 .

5. Let f and g be the functions satisfying $f(x)=(x)\sqrt{g(x)}$ for all real numbers x. If $y=4x-3$ is the equation of the tangent to the graph of g(x) at x=3, what is the equation of the line tangent to the graph of f(x) at x=3.

6. Determine the equation of the line **normal** to the curve $h(x)=\dfrac{(x+1)(2x-3)}{1-x}$ at $x=-1$.

7. Determine the point(s) (a,b) on the curve of $y=\dfrac{x+1}{x-2}$ where the slope of the tangent line is -3.

8. Line $y=k$ is tangent to the curve $f(x)=\dfrac{x^2-5}{x-(k+1)}$ at x =1. Find value of k.

MHF4UZ Mid-Review-Solution Date: ________________

1. Let f and g be differentiable functions such that $g(1)=3$, $f(1)=-2$ and $g'(1)=2$. Let $h(x)=\dfrac{fg}{f+g}$. If $h'(1)=5$, find the value of $f'(1)$.

$$h'(x)=\frac{(f(x)g(x))'(f(x)+g(x))-(f(x)+g(x))'(f(x)g(x))}{(f(x)+g(x))^2}$$

$$h'(x)=\frac{(f'(x)g(x)+f(x)g'(x))(f(x)+g(x))-(f'(x)+g'(x))(f(x)g(x))}{(f(x)+g(x))^2}$$

$$h'(1)=\frac{(f'(1)g(1)+f(1)g'(1))(f(1)+g(1))-(f'(1)+g'(1))(f(1)g(1))}{(f(1)+g(1))^2}$$

$$5=3f'(1)-4+6f'(1)+12$$

$$-3=9f'(1)$$

$$f'(1)=\frac{-1}{3}$$

2. Find the points on the curve $y=\left(1-\dfrac{x}{5}\right)^3$ where the slope of normal line is 15.

$$m_{\perp}=15\Rightarrow m_T=-\frac{1}{15}$$

$$y'=3\left(1-\frac{x}{5}\right)^2\left(\frac{-1}{5}\right)$$

$$-\frac{1}{15}=-\frac{3}{15}\left(1-\frac{x}{5}\right)^2$$

$$\frac{1}{9}=\left(1-\frac{x}{5}\right)^2$$

$$\pm\frac{1}{3}=1-\frac{x}{5}$$

$$\frac{1}{3}=1-\frac{x}{5} \quad \text{or} \quad -\frac{1}{3}=1-\frac{x}{5}$$

$$\frac{x}{5}=1-\frac{1}{3} \qquad \frac{x}{5}=1+\frac{1}{3}$$

$$\frac{x}{5}=\frac{2}{3} \qquad \frac{x}{5}=\frac{4}{3}$$

$$x=\frac{10}{3}\rightarrow y=\frac{1}{27} \qquad x=\frac{20}{3}\rightarrow y=-\frac{1}{27}$$

3. If $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}=-4$ and $g(x)=f\left(\sqrt{5-x^2}\right)$, determine the value of $g'(1)$.

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=-4\to f'(2)=-4$$

$$g(x)=f\left(\sqrt{5-x^2}\right)$$

$$g'(x)=f'\left(\sqrt{5-x^2}\right)\bullet\frac{-2x}{2\sqrt{5-x^2}}$$

$$g'(1)=f'(2)\bullet\frac{-1}{2}$$

$$=(-4)\left(\frac{-1}{2}\right)$$

$$=2$$

4. Assume that $h(x)=\left[f(x)\right]^3.g(x)$, where f and g are differentiable functions.

If $f(0)=\dfrac{-1}{2}, f'(0)=\dfrac{-8}{3}$ and $g(0)=-1, g'(0)=-2$, determine an equation of the line tangent to the graph of h at x=0 .

$$h'(x)=3\left[f(x)\right]^2.f'(x).g(x)+g'(x).\left[f(x)\right]^3$$

$$h'(0)=3\left[f(0)\right]^2.f'(0).g(0)+g'(0).\left[f(0)\right]^3$$

$$=3\left(\frac{-1}{2}\right)^2\left(\frac{-8}{3}\right)(-1)+(-2)\left(\frac{-1}{2}\right)^3$$

$$=2+\frac{1}{4}$$

$$=\frac{9}{4}$$

$$h(0)=\left[f(0)\right]^3 g(0)$$

$$=\left(\frac{-1}{2}\right)^3(-1)$$

$$=\frac{1}{8}$$

$$y-\frac{1}{8}=\frac{9}{4}x$$

5. Let f and g be the functions satisfying $f(x)=(x)\sqrt{g(x)}$ for all real numbers x. If $y=4x-3$ is the equation of the tangent to the graph of g(x) at x=3, what is the equation of the line tangent to the graph of f(x) at x=3.

$$f(x)=x\sqrt{g(x)}$$

$$f'(x)=\sqrt{g(x)}+\frac{g'(x)}{2\sqrt{g(x)}}\bullet(x)$$

$$f'(3)=\sqrt{g(3)}+\frac{g'(3)}{2\sqrt{g(3)}}\bullet(3)$$

$$f'(3)=\sqrt{9}+\frac{4}{2\sqrt{9}}\bullet(3)$$

$$m_T=5$$

Line $y=4x-3$ is the tangent to $g(x)$ at $x=3$, that means $g'(3)=4$.

Also $g(3)=4(3)-3=9$

$$f(3)=3\sqrt{g(3)}$$
$$=9$$

$$\therefore y-9=5(x-3)$$

6. Determine the equation of the line **normal** to the curve $h(x)=\frac{(x+1)(2x-3)}{1-x}$ at $x=-1$.

$$h(x)=\frac{2x^2-x-3}{1-x}\rightarrow h(-1)=0$$

$$h'(x)=\frac{(4x-1)(1-x)-(-1)(2x^2-x-3)}{(1-x)^2}$$

$$m_T=h'(-1)\rightarrow m_T=\frac{2}{5}$$

$$\therefore y=\frac{2}{5}(x+1)$$

7. Determine the point(s) (a,b) on the curve of $y=\frac{x+1}{x-2}$ where the slope of the tangent line is -3.

$$y'=\frac{-3}{(x-2)^2}$$

$$-3=\frac{-3}{(a-2)^2}\rightarrow(a-2)^2=1$$

$$a-2=\pm1$$
$$a=3 \text{ or } a=1$$
$$b=4 \text{ or } b=-2$$

8. Line $y = k$ is tangent to the curve $f(x) = \dfrac{x^2 - 5}{x - (k+1)}$ at x=1. Find value of k.

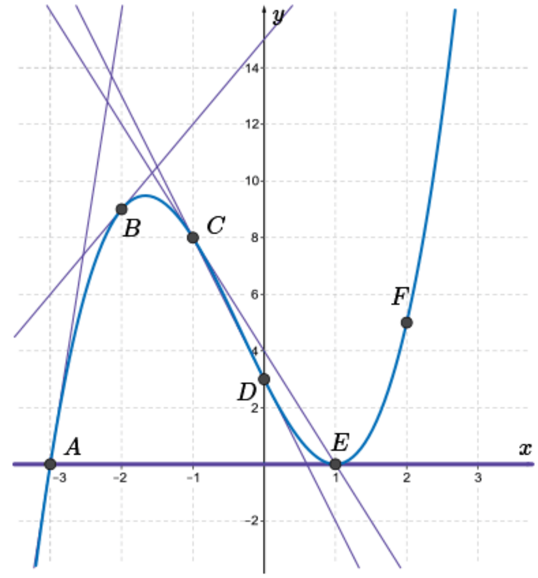
$$f'(x) = \frac{2x(x-k-1) - x^2 + 5}{(x-k-1)^2}$$

$$f'(1) = 0$$

$$2(1-k-1) - 1 + 5 = 0$$

$$-2k = -4$$

$$\boxed{k = 2}$$



So far, we have seen that the value of the derivative, $f'(x)$, gives us the instantaneous rate of change of a function, $f(x)$, at a point. It is represented graphically by the slope of the tangent line to the curve, $y = f(x)$, at that point. Throughout the graph, the slope of the tangent line is continually changing. We can describe this change as the rate of change of the slope of the tangent. To determine how the slope of the tangent is changing, we differentiate the derivative function f′(x). If f′(x) is differentiable, then the derivative of the derivative function can be found.

This is known as the **second derivative of** f(x), and is denoted in function notation as $f''(x)$.

In Leibniz notation, the second derivative is denoted as

$$\frac{d^2y}{dx^2} = \frac{d^2[f(x)]}{dx^2}.$$

Example 1

Find the second derivative of $f(x) = x^4 + 3x^2 - 5\sqrt{x}$

$$f'(x) = 4x^3 + 6x - \frac{5}{2}x^{-\frac{1}{2}}$$

$$f''(x) = 12x^2 + 6 + \frac{5}{4x^{\frac{3}{2}}}$$

Example 2: Find $\frac{d^2y}{dx^2}$, given $y = \frac{5x - 3}{2x}$. $\Rightarrow y = \frac{5}{2} - \frac{3}{2}x^{-1}$

$$\frac{dy}{dx} = \frac{3}{2}x^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{-3}{x^3}$$

Third Derivatives

The second derivative of f(x) is found by taking the derivative of f(x) twice. This can be extended further if f″(x) is differentiable; taking the derivative of f″(x) gives the third derivative of f(x), which is denoted in function notation as f‴(x) or $f^{(3)}(x)$.

In Leibniz notation, the third derivative is denoted as shown.

$$\frac{d^3y}{dx^3} = \frac{d^3[f(x)]}{dx^3}$$

Note that the brackets around the 3 are required in $f^{(3)}(x)$.

In general, if the derivatives remain differentiable, the n^{th} derivative of f(x) is found by taking its derivative n times, and is denoted $f^{(n)}(x)$.

Example 3: Determine the third derivative of the following functions

a) $y = \frac{1}{x}$ or $y = x^{-1}$

$y' = -x^{-2}$

$y'' = 2x^{-3}$

$y''' = -6x^{-4}$

$= \frac{-6}{x^4}$

b) $y = \frac{3}{2x-6}$ or $y = 3(2x-6)^{-1}$

$y' = -3(2x-6)^{-2}(2)$

$y' = -6(2x-6)^{-2}$

$y'' = 12(2x-6)^{-3}(2)$

$y'' = 24(2x-6)^{-3}$

$y''' = -72(2x-6)^{-4}(2)$

$y''' = \frac{-144}{(2x-6)^4}$

c) $y = \sqrt{x}$ or $y = x^{\frac{1}{2}}$

$y' = \frac{1}{2}x^{-\frac{1}{2}}$

$y'' = -\frac{1}{4}x^{-\frac{3}{2}}$

$y''' = \frac{3}{8}x^{-\frac{5}{2}}$

$y''' = \frac{3}{8x^{\frac{5}{2}}}$

Example 4: Suppose $f(x)=ax^2+bx+c$ and $f(1)=8, f'(1)=3$, and $f''(1)=-4$. Determine a,b, and c.

$f'(x) = 2ax + b$

$f'(1) = 3$

$3 = 2a + b$ — ①

$f''(x) = 2a$

$f''(1) = -4$

$-4 = 2a$

$\boxed{a = -2}$

Sub $a = -2$ in ①

$3 = 2(-2) + b$

$\boxed{b = 7}$

$\therefore f(x) = -2x^2 + 7x + c$

$f(1) = 8$

$8 = -2(1)^2 + 7(1) + c$

$8 = 5 + c$

$\boxed{c = 3}$

APPLICATIONS OF HIGHER ORDER DERIVATIVES – LINEAR MOTION

Definitions

Position s(t) is the location of an object at a value of time t.

Velocity v(t) is the rate of change of position over time, so

$$\boxed{v(t) = s'(t) = \frac{ds}{dt}}$$

Acceleration a(t) is the rate of change of velocity over time, so

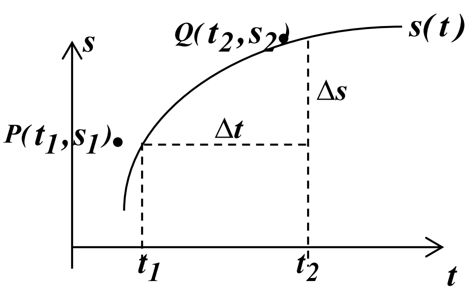
$$\boxed{a(t) = v'(t) = s''(t)}$$

Or

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Note: These are vector quantities with a magnitude and direction.

Position



- $s(0)$ represents the initial position of the object (when $t = 0$)
- $s(t) > 0$ indicates that the object is to the right of the origin
- $s(t) < 0$ indicates that the object is to the left of the origin.
- $s(t) = 0$ indicates that the object is at the origin (where it started)

Velocity

Thus instantaneous velocity is $v(t) = s'(t) = \dfrac{ds}{dt}$

- $v(0)$ is the initial velocity (when $t = 0$)
- $v(t) > 0$ $\Rightarrow$ object is moving to the right (positive direction)
- $v(t) < 0$ $\Rightarrow$ object is moving to the left (negative direction)
- $v(t) = 0$ $\Rightarrow$ object is at rest or object may be changing directions or object may be at a maximum/minimum height

momentarily stops

Acceleration

- $a(t) > 0$ $\Rightarrow$ object is accelerating (velocity is increasing)
- $a(t) < 0$ $\Rightarrow$ object is decelerating (velocity is decreasing)
- $a(t) = 0$ $\Rightarrow$ object is at a constant velocity or object is at a max/min velocity

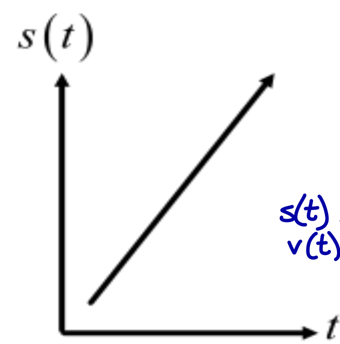
(ie. cruise control)

If $sv > 0$ the object is moving away from the origin → same directions/signs ++ or - -
If $sv < 0$ the object is moving towards the origin → opposite directions/signs +/-

If $av > 0$ the object is speeding up → same directions/signs ++ - -
If $av < 0$ the object is slowing down → opposite directions/signs +/-

Example 5: In each graph determine if the object is moving away from/towards the origin or speeding up/slowing down

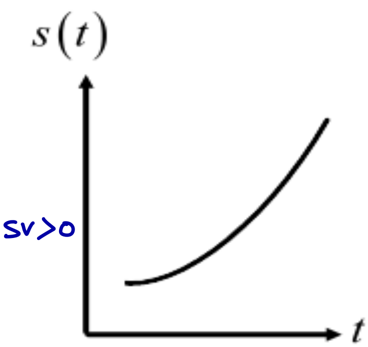
(a)



$s(t) > 0$ $\Rightarrow sv > 0$
$v(t) > 0$

Moving away from origin

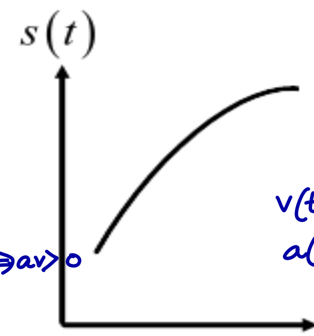
(b)



$v(t) > 0$ $\Rightarrow av > 0$
$a(t) > 0$

Moving away from origin, Speeding up

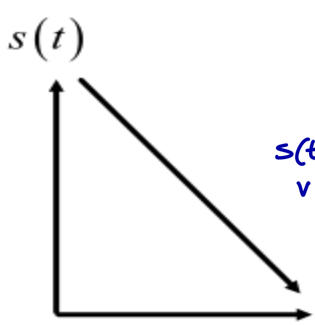
(c)



$v(t) > 0$ $\Rightarrow av < 0$
$a(t) < 0$

Moving away from origin, Slowing down

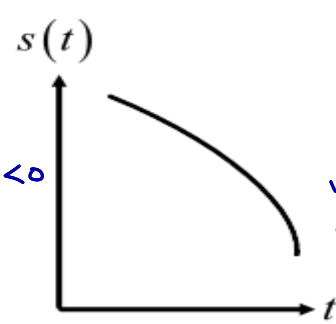
(d)



$s(t) > 0$ $\Rightarrow sv < 0$
$v(t) < 0$

Moving toward the origin

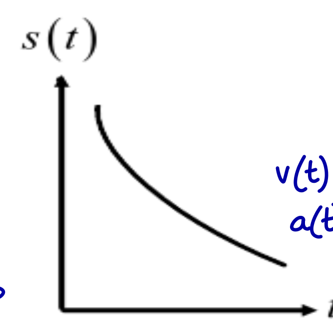
(e)



$v(t) < 0$
$a(t) < 0$
$\Downarrow$
$av > 0$

Moving toward the origin, Speeding up

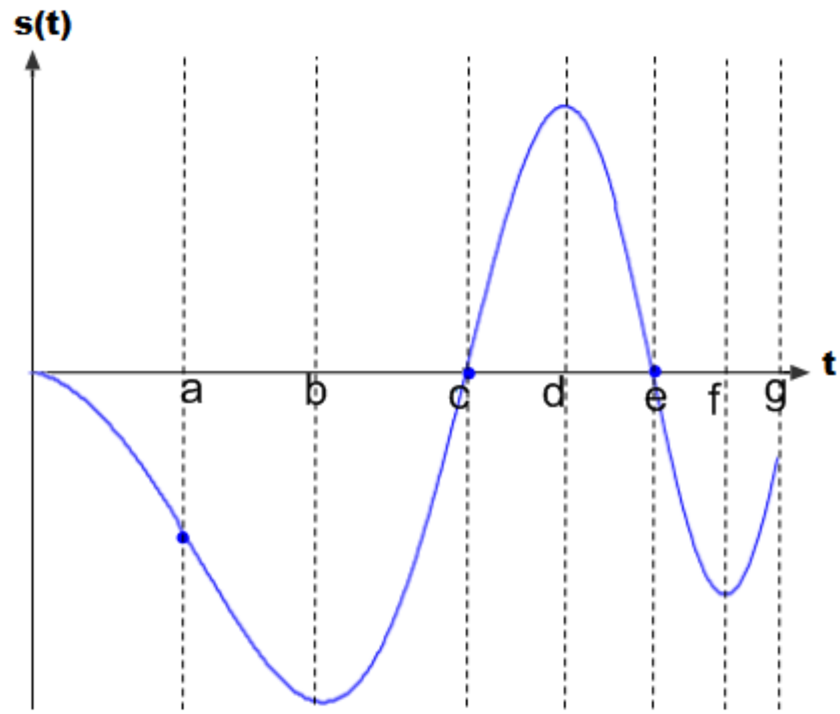
(f)



$v(t) < 0$
$a(t) > 0$
$\Downarrow$
$av < 0$

Moving toward the origin, Slowing down

Example 6: Complete the table below the graph.



Interval	Displacement +/-	Velocity +/-	Acceleration +/-	$a(t)$ $v(t)$ Speeding up/ Slowing down	$s(t)$ $v(t)$ Moving away/ toward
[0,a]	–	–	–	Speeding up	Moving away
[a,b]	–	–	+	slowing down	Moving away
[b,c]	–	+	+	Speeding up	Moving toward
[c,d]	+	+	–	slowing down	Moving away
[d,e]	+	–	–	Speeding up	Moving toward
[e,f]	–	–	+	slowing down	Moving away
[f,g]	–	+	+	Speeding up	Moving toward

Example 7: An object moves in a straight line according to the motion equation

$s(t)=2t^3-15t^2+36t-22\,,t\geq 0$. where t is in seconds and $s(t)$ is in metres

(a) Find the velocity and acceleration at time ***t***.

$s'(t)=v(t)=6t^2-30t+36\ ,t\geq 0$

$s''(t)=v'(t)=a(t)=12t-30\ ,t\geq 0$

(b) Find the initial conditions and interpret their meaning.

At $t=0$, $s(0)=-22$, $v(0)=36$, $a(0)=-30$

The object is 22 m to the left of the origin and the initial velocity is 36 m/s to the right.

The object initially decelerate at a rate of 30 m/s².

(c) Find the average velocity and acceleration from ***t*** = 2 to ***t*** = 4.

Avg velocity

$=\dfrac{s(4)-s(2)}{4-2}$

$=\dfrac{[2(4)^3-15(4)^2+36(4)-22]-[2(2)^3-15(2)^2+36(2)-22]}{2}$

$=\dfrac{10-6}{2}$

$=2$ Average velocity is 2 m/s

Avg accelation $=\dfrac{v(4)-v(2)}{4-2}$

$=\dfrac{[6(4)^2-30(4)+36]-[6(2)^2-30(2)+36]}{2}$

$=\dfrac{12-0}{2}$

$=6$

Average acceleration is 6 m/s²

(d) When is the particle at rest?

Particle is at rest when $v(t)=0$

$6t^2-30t+36=0$

$6(t^2-5t+6)=0$

$6(t-3)(t-2)=0$

$t=3$ or $t=2$

The particle is at rest when $t=2$ s and $t=3$ s

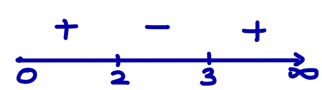
(e) When does the object move in a positive direction?

$v(t)>0$

$6(t-3)(t-2)>0$

Method #1: Graphing

Method #2: Interval Chart



Note leading coefficient is positive ⇒ start with positive sign on the far right. Sign changes if the degree of factor is odd, sign remains the same if the degree is even.

Object moves in a positive direction when

$0\leq t<2$ and $t>3$

Example 8: The motion of a particle on straight line is given by position function $s(t) = 36 - 24t + 9t^2 - t^3$, where **s** is in meter and **t** is in minute.

$s(t) = 36 - 24t + 9t^2 - t^3$
$= -(t^3 - 9t^2 + 24t - 36) \Rightarrow s(6) = 0$

$t-6$ is a factor of $s(t)$

$$\begin{array}{r|rrrr} -6 & 1 & -9 & 24 & -36 \\ & & -6 & 18 & -36 \\ \hline & 1 & -3 & 6 & 0 \end{array}$$

$\therefore s(t) = -(t-6)(t^2-3t+6)$

(a) Find the velocity and acceleration at time **t**.

$v(t) = -24 + 18t - 3t^2$
$v(t) = -3(t^2 - 6t + 8)$
$= -3(t-4)(t-2)$

$a(t) = 18 - 6t$
$a(t) = -6(t-3)$

(b) After how many minutes does the object stop?

$v(t) = 0$ when the object stops

$\therefore \; 0 = -3(t-4)(t-2)$

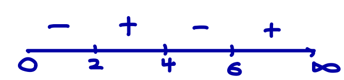
$\therefore t = 4$ or $t = 2$

The object stops after 2 min and 4 min

(c) When is the particle moving toward the motion detector?

$s(t) \cdot v(t) < 0$ when object moves towards the reference point

$s(t) = -(t-6)(t^2-3t+6)$
$v(t) = -3(t-4)(t-2)$
$\Rightarrow s(t)v(t) = 3(t-6)(t^2-3t+6)(t-4)(t-2)$

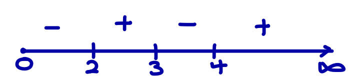


$\therefore$ The object is moving toward the detector when $0 \le t < 2$ and $4 < t < 6$

(d) When is the object slowing down?

$v(t) \cdot a(t) < 0$ when object is slowing down

$v(t) = -3(t-4)(t-2)$
$a(t) = -6(t-3)$
$a(t)\,v(t) = 18(t-2)(t-3)(t-4)$



Object is slowing down when $0 \le t < 2$ and $3 < t < 4$

(e) Determine the total distance traveled in the first 7 minutes.

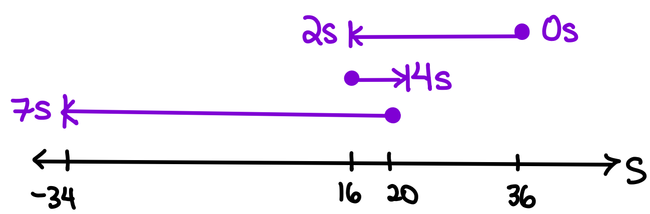
From b) the object stops at $t=2$ and $t=4$ and considering the interval $[0, 7]$.
Find the position of the object at these times.

$s(0) = 36$

$s(2) = 36 - 24(2) + 9(2)^2 - (2)^3$
$= 16$

$s(4) = 36 - 24(4) + 9(4)^2 - (4)^3$
$= 20$

$s(7) = 36 - 24(7) + 9(7)^2 - (7)^3$
$= -34$



Total distance $= |s(7) - s(4)| + |s(4) - s(2)| + |s(2) - s(0)|$
$= |-34 - 20| + |20 - 16| + |16 - 36|$
$= 54 + 4 + 20$
$= 78$

Use GDC
1) Enter function in Y_1
2) Vars → Y-vars → Function → Y_1 → Enter $\boxed{(}\ \boxed{7}\ \boxed{)}$ → $\boxed{\text{Enter}}$

OR

1) Graph the function
2) Press $\boxed{\text{TRACE}}$ then enter the t-value

Application of Higher Order Derivatives: Linear Motion

	Displacement	Velocity	Acceleration
Notation:	$s(t)$	$v(t)$	$a(t)$
Unit of measure:	m	m/s or ms^{-1}	m/s^2 or ms^{-2}
Order of Derivative	n/a	$s'(t)$ or $\frac{ds}{dt}$	$s''(t)$ or $\frac{d^2s}{dt^2}$ or $v'(t)$
Definition:	The change of position ↳ Δs	Rate of change of position with respect to time ↳ $\frac{\Delta s}{\Delta t}$	The rate of change of velocity with respect to time ↳ $\frac{\Delta v}{\Delta t}$
Other notes:	Vector quantity (with magnitude & direction) vs. distance (scalar)	Vector quantity vs. speed (scalar)	Vector quantity

1. Vertical Motion:

A ball is thrown up and its motion is described by $h(t) = -4.9t^2 + 6t + 2$ where h is the height in metres and t is the time in seconds.

(a) Find the velocity and acceleration function:

$v(t) = -9.8t + 6$ ← $h'(t)$

$a(t) = -9.8$ ← $v'(t)$ or $h''(t)$

(b) Find the initial velocity, v_0.

$v(t) = -9.8t + 6$

$v(0) = -9.8(0) + 6$

$= 6$

∴ The initial velocity is 6m/s upwards.

(c) When does the ball reach its max height? What is this height?

At max height, $v(t) = 0$ (momentarily stops)

$v(t) = -9.8t + 6$

$0 = -9.8t + 6$

$\frac{-6}{-9.8} = t$

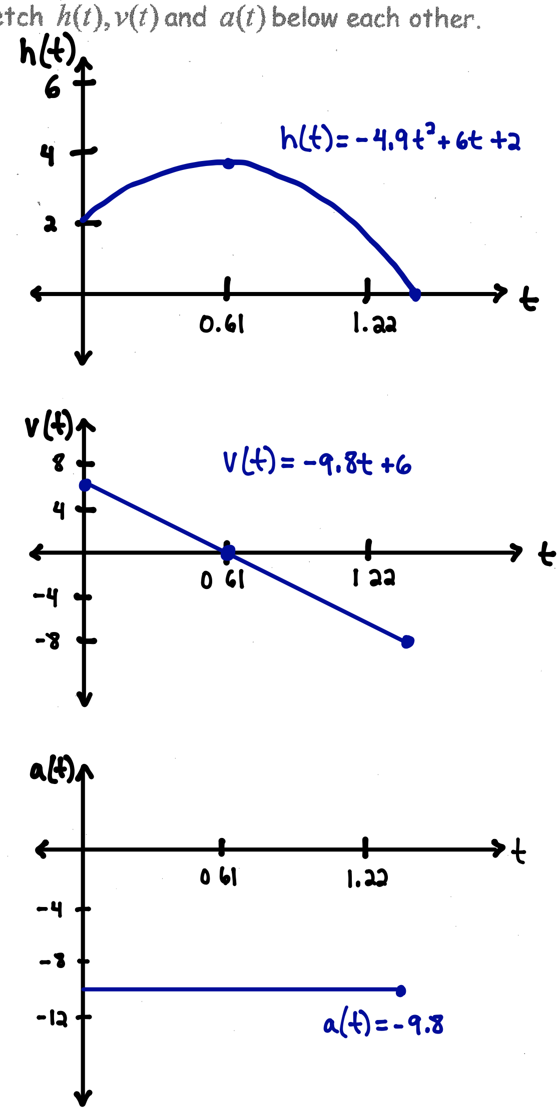
$0.612 = t$

$h(0.612) = -4.9(0.612)^2 + 6(0.612) + 2$

$\doteq 3.84$

∴ The ball reaches its max height of approx 3.84m at approx. 0.612s.

(d) Sketch $h(t), v(t)$ and $a(t)$ below each other.



(e) When does the ball hit the ground?

Ball hits the ground when $h(t)=0$:

$\therefore 0=-4.9t^2+6t+2$

$t=\dfrac{-(6)\pm\sqrt{(6)^2-4(-4.9)(2)}}{2(-4.9)}$

$=\dfrac{6\pm\sqrt{75.2}}{9.8}$

$\therefore t \doteq 1.50$ or $t \doteq -0.273$ ← inadmissible since $t>0$

$\therefore$ The ball will hit the ground at approx. 1.50 s.

(f) What is the velocity (rate of change of height) upon impact?

$v(t)=-9.8t+6$

$v(1.50) \doteq -9.8(1.50)+6$

$\doteq -8.7$

$\therefore$ The velocity upon impact is approx. 8.7 m/s downwards.

(g) What is the acceleration (rate of change of velocity) upon impact?

$a(t)=-9.8$

$a(1.50)=-9.8$

$\therefore$ The acceleration upon impact is 9.8 m/s² downwards.

(h) Determine the time interval when the ball is speeding up ([redacted]) and when it is slowing down

If $v(t)\cdot a(t)>0$ → the object is speeding up

If $v(t)\cdot a(t)<0$ → the object is slowing down

If $v(t)\cdot a(t)=0$ → the object is neither speeding up or slowing down (constant speed) ie. cruise control or stopped

Note: for vertical motion, acceleration is always negative (due to gravity), hence it depends on the sign of the velocity.

$\therefore$ The ball is slowing down in the interval $0\leq t<0.612, t\in\mathbb{R}$ when it is going up.

$\therefore$ The ball is speeding up in the interval $0.612<t\leq 1.50, t\in\mathbb{R}$ when it is coming down.

Exit Card!

Given position function $s(t)=t^{\frac{5}{2}}(7-t),\ t\geq 0$.

$s(t)=7t^{\frac{5}{2}}-t^{\frac{7}{2}}$

(a) Is the particle speeding up or slowing down when $t=4s$.

$v(t)=\frac{35}{2}t^{\frac{3}{2}}-\frac{7}{2}t^{\frac{5}{2}},\ t\geq 0$

$=\frac{7}{2}t^{\frac{3}{2}}(5-t)$

$v(4)>0$

$a(t)=\frac{105}{4}t^{\frac{1}{2}}-\frac{35}{4}t^{\frac{3}{2}}$

$=\frac{35}{4}t^{\frac{1}{2}}(3-t)$

$a(4)<0$

$a(4)\cdot v(4)<0$

Object is slowing down at $t=4s$

(b) At what time(s) is the object at rest?

$v(t)=\frac{7}{2}t^{\frac{3}{2}}(5-t)$

$v(t)=0$

$0=\frac{7}{2}t^{\frac{3}{2}}(5-t)$

$t=0$ or $t=5$

The object is at rest at $t=0s$ and $t=5s$

(c) In which direction is the object moving at ***t* =4**

$v(t)=\frac{7}{2}t^{\frac{3}{2}}(5-t)$

+ −

0 5 ∞

The object is moving in a positive direction at $t=4$

(d) When is the object moving in a negative direction?

$v(t)=\frac{7}{2}t^{\frac{3}{2}}(5-t)$

+ −

0 5 ∞

The object is moving in a negative direction when $t>5s$

(e) When does the object return to its initial position?

$s(t)=0$

$s(t)=7t^{\frac{5}{2}}-t^{\frac{7}{2}}$

$0=t^{\frac{5}{2}}(7-t)$

$t=0$ or $t=7$

The object return to its initial position at $7s$

MHF4UE Practice: Higher Order Derivatives

1. Find the first and second derivatives of each function.

a) $f(x) = 2x^4 - 4x^{-2}$

$f'(x) = 8x^3 + 8x^{-3}$

$f''(x) = 24x^2 - 24x^{-4}$

b) $y = \dfrac{3}{x^2}$

$f(x) = 3x^{-2}$

$f'(x) = -6x^{-3} \rightarrow f'(x) = \dfrac{-6}{x^3}$

$f''(x) = 18x^{-4} \rightarrow f'(x) = \dfrac{18}{x^4}$

c) $y = \dfrac{2x+1}{x}$

$f(x) = 2 + x^{-1}$

$f'(x) = -x^{-2} \rightarrow f'(x) = \dfrac{-1}{x^2}$

$f''(x) = 2x^{-3} \rightarrow f'(x) = \dfrac{2}{x^3}$

d) $g(x) = (x - 1)(x + 1)^3$

$g'(x) = (x+1)^3 + 3(x+1)^2(x-1)$

$g''(x) = 3(x+1)^2 + 6(x+1)(x-1) + 3(x+1)^2$

$= 6(x+1)^2 + 6(x^2 - 1)$

$= 12x(x+1)$

g) $y = \dfrac{x^2 - 4}{x+1}$

$y' = \dfrac{2x(x+1) - (x^2 - 4)}{(x+1)^2}$

$y' = \dfrac{x^2 + 2x + 4}{(x+1)^2}$

$y'' = \dfrac{(2x+2)(x+1)^2 - 2(x+1)(x^2 + 2x + 4)}{(x+1)^4}$

$= \dfrac{-6}{(x+1)^3}$

h) $s(t) = t^3 + \dfrac{2}{\sqrt{t}}$

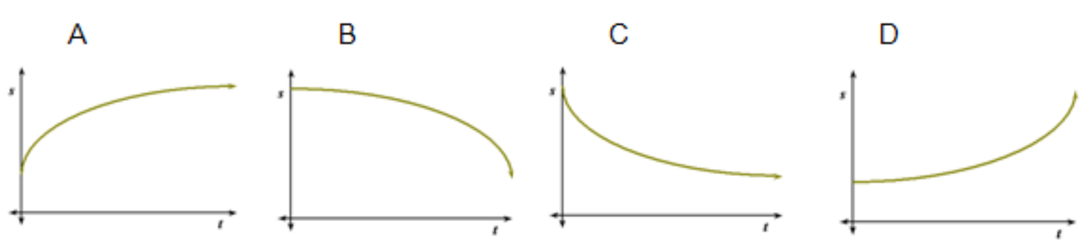
$s(t) = t^3 + 2t^{\frac{-1}{2}}$

$s'(t) = 3t^2 - t^{-\frac{3}{2}} \rightarrow s'(t) = 3t^2 - \dfrac{1}{t\sqrt{t}}$

$s''(t) = 6t + \dfrac{3}{2}t^{-\frac{5}{2}} \rightarrow s''(t) = 6t + \dfrac{3}{2t^2\sqrt{t}}$

2. A boat demonstrates a positive velocity but a negative acceleration. Which of the following plots illustrates its position?

A



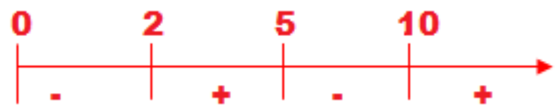
3. A particle moves on the y axis with this relationship between position and time: $s(t) = t^3 - 17t^2 + 80t - 100$ .Determine the time interval(s) during which it is :

a) located below the origin

$s(t) = (t-2)(t-5)(t-10)$

$s(t) < 0$

$t \in (0,2) \cup (5,10)$



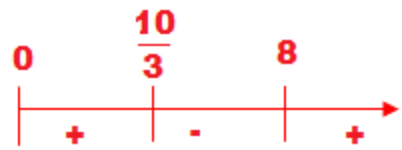
b) moving upward(moving in positive direction)

$v(t) = 3t^2 - 34t + 80$

$v(t) = (3t-10)(t-8)$

$v(t) > 0$

$t \in \left(0, \frac{10}{3}\right) \cup (8, \infty)$



c) slowing down

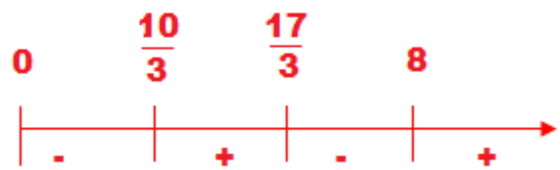
$v(t) = 3t^2 - 34t + 80$

$a(t) = 6t - 34$

$a(t)v(t) = 2(3t-17)(3t-10)(t-8)$

$a(t)v(t) < 0$

$t \in \left(0, \frac{10}{3}\right) \cup \left(\frac{17}{3}, 8\right)$



d) moving away from the origin

$v(t) = (3t-10)(t-8)$

$s(t) = (t-2)(t-5)(t-10)$

$s(t)v(t) = (t-2)(t-5)(t-10)(3t-10)(t-8)$

$s(t)v(t) > 0$ → s(t) and v(t) is in the same direction

$t \in \left(2, \frac{10}{3}\right) \cup (5,8) \cup (10, \infty)$

4. A particle moves on the y axis with this relationship between position and time:

$$s(t) = \frac{1}{4}t^4 - 2t^3 + \frac{9}{2}t^2 - 4t + 2$$

a) Describe the motion of the particle at $t = 0$.

$s(0) = 2$ The initial position of object is $2\,\text{m}$ ~~to the right of~~ above the origin

b) What is the average velocity of the particle between $t = 1$ and $t = 4$ seconds?

$$\textit{average velocity} = \frac{s(4) - s(1)}{4 - 1}$$

$$= \frac{-6 - \frac{3}{4}}{3}$$

$$= -2.35 \text{ m/s}$$

c) When does the particle reverse direction?

$v(t) = s'(t) = t^3 - 6t^2 + 9t - 4$

$v(t) = (t-4)(t-1)^2$

$v(t) = 0$

$t = 4s$

d) Find the total distance traveled from $t = 0$ to $t = 5$ seconds

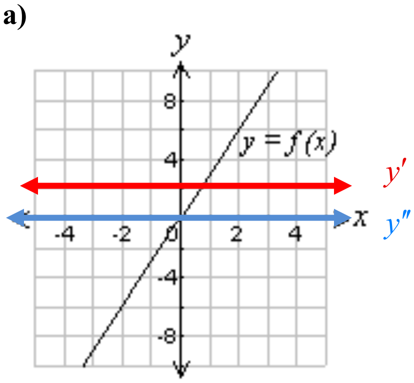
$$\text{Total distance} = |s(4) - s(0)| + |s(5) - s(4)|$$

$$= |-6 - 2| + \left|\frac{3}{4} - (-6)\right|$$

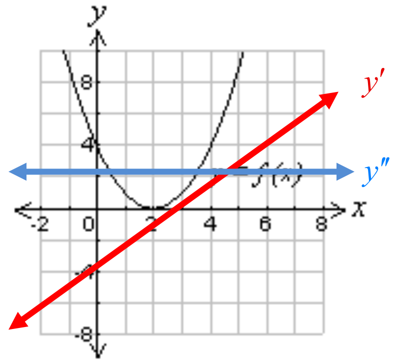
$$= \frac{59}{4} m$$

5. For each graph below, sketch the corresponding graphs of f' and f''.

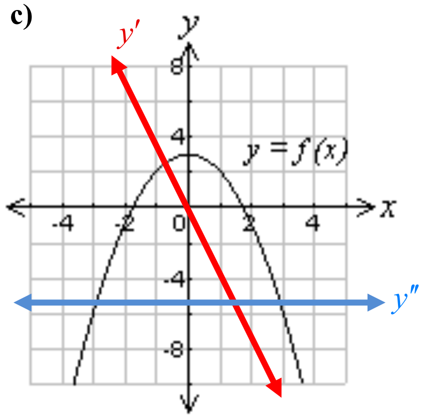
a)



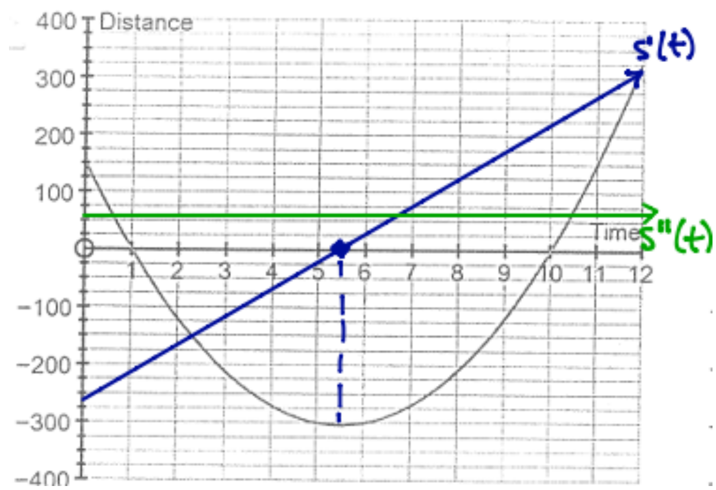
b)



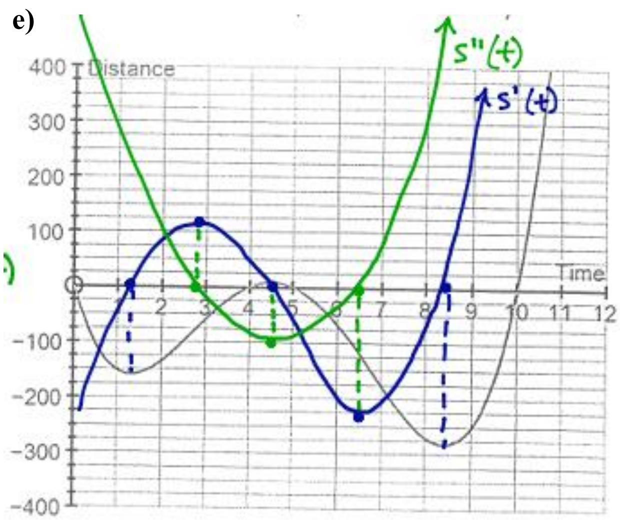
c)



d)



e)



6. For what values of the constants a, b, c, and d does the function $f(x) = ax^3 + bx^2 + cx + d$ satisfy both of the following conditions?

a) $f''(0) = 0$ at the origin **b)** a horizontal tangent at (2, 4)

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f(0) = 0 \rightarrow d = 0$

$f''(0) = 0 \rightarrow 2b = 0 \ \ or \ \ b = 0$

$f'(2) = 0 \rightarrow 3a(4) + c = 0 \rightarrow 12a + c = 0$

$f(2) = 4 \rightarrow 8a + 2c = 4 \rightarrow 4a + c = 2$

$$\begin{cases} 12a + c = 0 \\ 4a + c = 2 \end{cases} \Rightarrow a = \frac{-1}{4}, c = 3$$

7. A person's height, in metres, can be modelled by the function $h(t) = \dfrac{at}{b+t} + c$, where t is the age of the person, in years, and a, b, and c are positive constants.

a) $h'(t)$ represents the growth rate. What does $h''(t)$ represent?

$h''(t)$ represents how fast a person's height will change as the person gets older

b) Show that $h''(t)$ is always negative. What does this indicate about the growth rate?

$$h(t) = \frac{at}{b+t} + c$$

$$h'(t) = \frac{a(b+t) - at}{(b+t)^2}$$

$$h'(t) = \frac{ab}{(b+t)^2}$$

$$h'(t) = ab(b+t)^{-2}$$

$$h''(t) = -2ab(b+t)^{-3}$$

$$= \frac{-2ab}{(b+t)^3}$$

$$h''(t) < 0$$

It indicates the growth rate will decrease as person gets older

c) Show that

i) the initial height is c

$$h(t) = \frac{at}{b+t} + c$$

$$h(0) = c$$

ii) the initial growth rate is $\dfrac{a}{b}$

$$h'(t) = \frac{ab}{(b+t)^2}$$

$$h'(0) = \frac{ab}{b^2}$$

$$= \frac{a}{b}$$

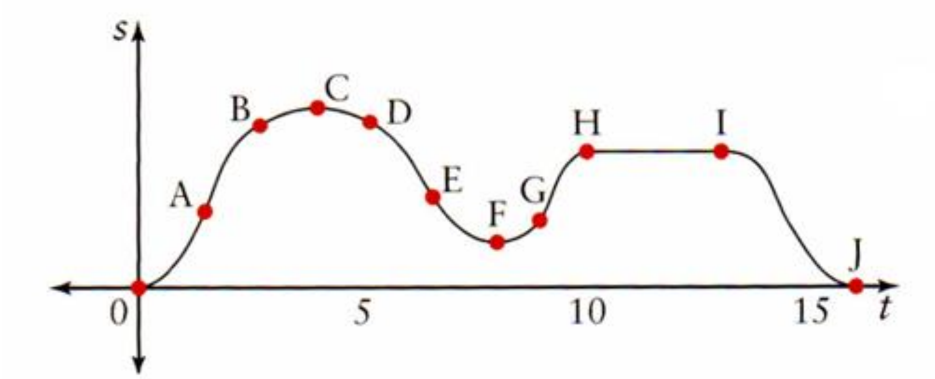
d) Suggest reasonable values for the constants.

Answers may vary

e) In what way(s) is the function not a realistic model for the height of a person?

After a certain age the height won't increase!

8. The following graph shows the position function of a bus during 15 min trip.



a) What is the initial velocity of the bus? 0

b) What is the bus's velocity at C and F? 0 and 0

c) Is the bus going faster at A or at B? Explain. A , the tangent line is steeper

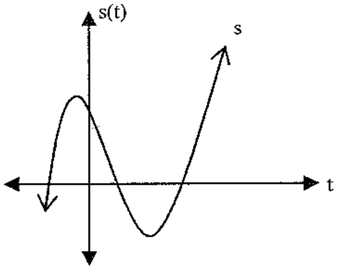
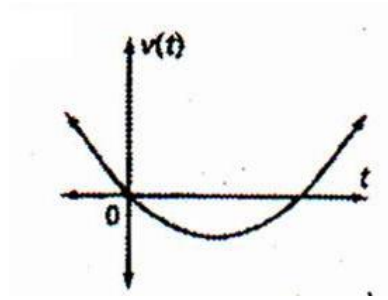
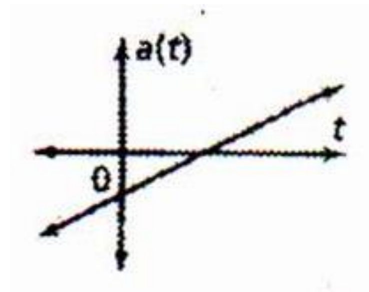
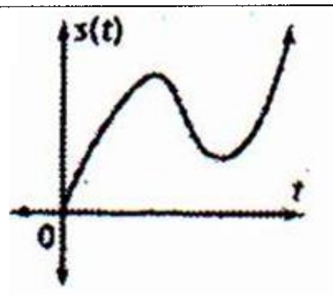
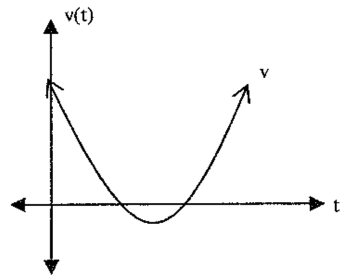
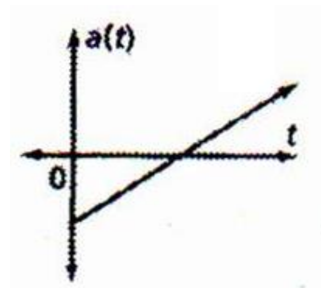
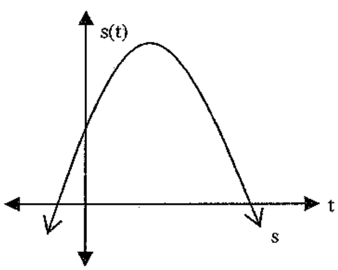
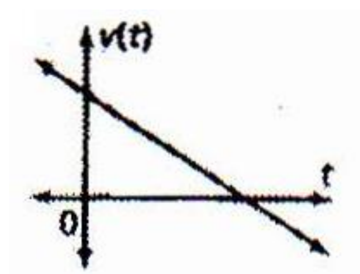
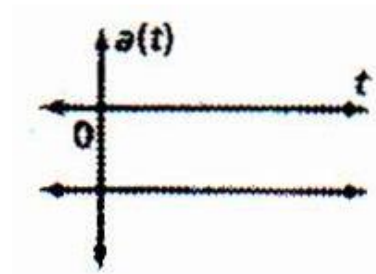
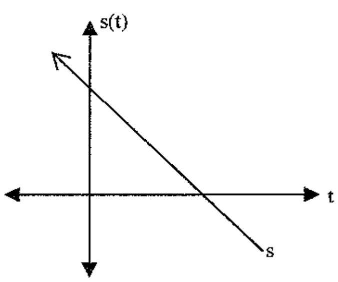
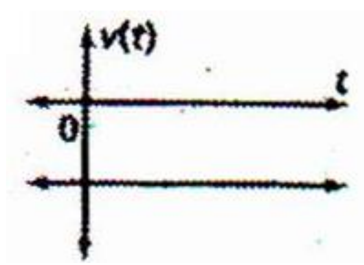
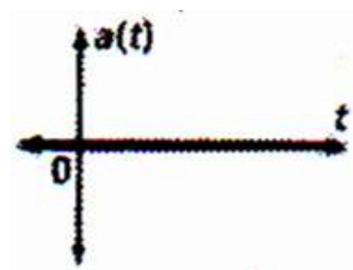
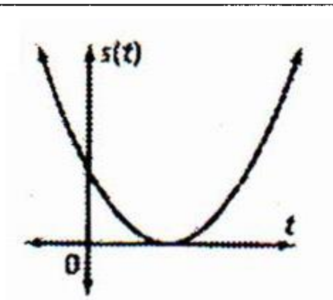
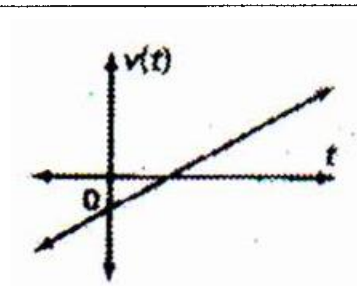
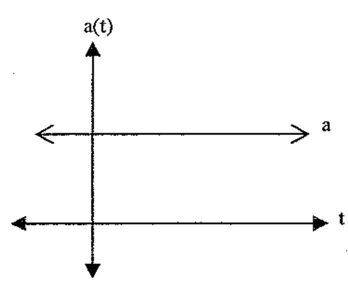
d) What happens to the motion of the bus between H and I? The bus is at rest

e) What happens at J? Bus returns to the origin

9. Refer to the graph in question 8. Is the acceleration positive, zero, or negative during the following intervals?

a) 0 to A	**b)** C to D	**c)** E to F	**d)** G to H	**e)** F to G
Positive	Negative	Positive	Negative	Positive

10. Create sketches so that each graph in a set corresponds to the other two.

	Displacement	**Velocity**	**Acceleration**
a)			
b)			
c)			
d)			
e)			

Warm Up

The forward motion of a space shuttle, t seconds after touchdown, is described by $s(t) = 189t - t^{\frac{7}{3}}$, where s is measured in metres.

a) How much time is needed for the shuttle to stop?

$v(t) = 189 - \frac{7}{3}t^{\frac{4}{3}}$

$0 = 189 - \frac{7}{3}t^{\frac{4}{3}}$

$\frac{7}{3}t^{\frac{4}{3}} = 189$

$t^{\frac{4}{3}} = 189\left(\frac{3}{7}\right)$

$t^{\frac{4}{3}} = 81$

$t^4 = 531441$

$t = 27, t \geq 0$

$\therefore$ The shuttle will need 27 s.

b) How far does the shuttle travel from touchdown to stop?

$s(t) = 189t - t^{\frac{7}{3}}$

$s(27) = 189(27) - 27^{\frac{7}{3}}$

$s(27) = 2916$

$\therefore$ The shuttle travel 2916 m from touchdown to stop.

c) Does the shuttle speeding up or slowing down at 8 seconds after touchdown?

$v(t) = 189 - \frac{7}{3}t^{\frac{4}{3}}$

$a(t) = -\frac{28}{9}t^{\frac{1}{3}}$

$v(8)a(8) = \left[189 - \frac{7}{3}(8)^{\frac{4}{3}}\right]\left[-\frac{28}{9}(8)^{\frac{1}{3}}\right]$

$= \left[189 - \frac{112}{3}\right]\left[-\frac{28}{9}(2)\right]$

$= \left(\frac{455}{3}\right)\left(-\frac{56}{9}\right)$

$= -\frac{25480}{27} < 0$

$\therefore$ The shuttle is slowing down at 8 s after touchdown.

Implicit Differentiation

Until now, we have described functions by expressing one variable explicitly in terms of another variable: $y = f(x)$. Example: $y = x^2$, $y = \sqrt{1-x^2}$, $y = x^3 - 3x$.

However, curves may be defined by a relation such as:

$$x^2 + y^2 = 9$$

$$\frac{x^2}{4} + y^2 = 1$$

$$y^5 + x^2y - 2x^2 = -1$$

Use a method called **implicit differentiation**.

Strategy: differentiate both sides of the equation with respect to x and then solve for $\frac{dy}{dx}$ or y'.

(Note: use chain rule when differentiating terms containing y.)

Recall: $\frac{dx^2}{dx} = 2x$ and $\frac{dh^2}{dh} = 2h$

However: $\frac{dy^2}{dx} = 2y\frac{dy}{dx}$

Ex. 1. Differentiate:

$$\frac{x^2}{4} + y^2 = 1$$

$$\frac{2x}{4} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{\frac{-2x}{4}}{2y}$$

$$= \frac{-2x}{4} \times \frac{1}{2y}$$

$$= \frac{-x}{4y}$$

Ex. 2. Differentiate:

$$y^5 + x^2y - 2x^2 = 1$$

$$5y\frac{dy}{dx} + 2xy + x^2\frac{dy}{dx} - 4x = 0$$

$$\frac{dy}{dx}(5y + x^2) = 4x - 2xy$$

$$\frac{dy}{dx} = \frac{4x - 2xy}{5y^4 + x^2}$$

Ex. 3. Find the equation of the tangent to the curve $3x^4 + x^2y^2 + y^3 = 5$ at the point (1, 1).

$$12x^3 + 2xy^2 + x^2 \cdot 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(2x^2y + 3y^2\right) = -12x^3 - 2xy^2$$

$$\frac{dy}{dx} = \frac{-12x^3 - 2xy^2}{2x^2y + 3y^2}$$

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-12(1)^3 - 2(1)(1)^2}{2(1)^2(1) + 3(1)^2}$$

$$= \frac{-14}{5}$$

$$y - (1) = \left(\frac{-14}{5}\right)[x - (1)]$$

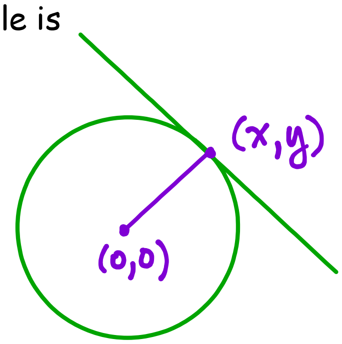
$$y - 1 = \frac{-14}{5}x + \frac{14}{5}$$

$$y = \frac{-14}{5}x + \frac{19}{5}$$

or $14x + 5y - 19 = 0$

$\therefore$ The equation of the tangent at $(1,1)$ is $y = \frac{-14}{5}x + \frac{19}{5}$.

Ex. 4. If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$. Verify that any tangent line to this circle is perpendicular to the radius at the point of tangency.



$$x^2 + y^2 = 25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$m_{radius} = \frac{y - 0}{x - 0}$$

$$= \frac{y}{x}$$

$$\text{slope of tangent} \times m_{radius} = \left(-\frac{x}{y}\right)\left(\frac{y}{x}\right)$$

$$= -1$$

$\because$ The slope of the tangent and the slope of the radius at the point of tangency are negative reciprocals,

$\therefore$ they are perpendicular.

Ex. 5. Find the slope of the tangent line to the curve $y = \sqrt[3]{6+x^2}$ at the point $(\sqrt{2}, 2)$.

Method ①: Implicit

$y^3 = 6 + x^2$

$3y^2 \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{3y^2}$

$\left.\frac{dy}{dx}\right|_{(\sqrt{2},2)} = \frac{2(\sqrt{2})}{3(2)^2}$

$= \frac{\sqrt{2}}{6}$

Method ②: Explicit

$y = (6+x^2)^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}(6+x^2)^{-\frac{2}{3}}(2x)$

$= \frac{2x}{3(6+x^2)^{\frac{2}{3}}}$

$\left.\frac{dy}{dx}\right|_{(\sqrt{2},2)} = \frac{2(\sqrt{2})}{3[6+(\sqrt{2})^2]^{\frac{2}{3}}}$

$= \frac{2\sqrt{2}}{3(4)} \rightarrow = \frac{\sqrt{2}}{6}$

Ex. 6. Find the equation of the tangent to the curve $\sqrt{y} = \sqrt{xy} - 2\sqrt{x}$ at $x = 4$.

1) $y^{\frac{1}{2}} = (xy)^{\frac{1}{2}} - 2x^{\frac{1}{2}}$

$\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = \frac{1}{2}(xy)^{-\frac{1}{2}}\left(y + x\frac{dy}{dx}\right) - x^{-\frac{1}{2}}$

$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = \frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}}\frac{dy}{dx} - \frac{1}{\sqrt{x}}$

$\frac{dy}{dx}\left(\frac{1}{2\sqrt{y}} - \frac{x}{2\sqrt{xy}}\right) = \frac{y}{2\sqrt{xy}} - \frac{1}{\sqrt{x}}$

$\frac{dy}{dx} = \frac{y - 2\sqrt{y}}{2\sqrt{xy}} \div \frac{\sqrt{x} - x}{2\sqrt{xy}}$

$= \frac{y - 2\sqrt{y}}{\sqrt{x} - x}$

2) At $x = 4$,

$\sqrt{y} = \sqrt{(4)y} - 2\sqrt{(4)}$

$\sqrt{y} = 2\sqrt{y} - 4$

$4 = \sqrt{y}$

$16 = y$ $\quad \therefore$ Tangent point is $(4, 16)$

3) $\left.\frac{dy}{dx}\right|_{(4,16)} = \frac{(16) - 2\sqrt{16}}{\sqrt{(4)} - (4)}$

$= -4$

4) $y - (16) = (-4)[x - (4)]$

$y - 16 = -4x + 16$

$y = -4x + 32$

$\therefore$ The equation of the tangent at $x = 4$ is $y = -4x + 32$.

Implicit Differentiation

Explicit form	**Implicit form**
$y = \dfrac{2}{x}$	$xy = 2$

Example 1:

Find the derivative of both $y = \dfrac{2}{x}$ and $xy = 2$ using explicit and implicit differentiation, respectively, and show that the two have equivalent derivative functions $\dfrac{dy}{dx}$.

$y = 2x^{-1}$

$\dfrac{dy}{dx} = \dfrac{-2}{x^2}$

$xy = 2$

$1(y) + x\dfrac{dy}{dx} = 0$

$x\dfrac{dy}{dx} = -y$

$x\dfrac{dy}{dx} = -\dfrac{2}{x}$

$\dfrac{dy}{dx} = \dfrac{-2}{x^2}$

Most of the time, we use implicit differentiation when we're dealing with curves that are not functions, with powers of y greater than 1. This is why we cannot solve for y explicitly. We will, however, be able to solve for $\dfrac{dy}{dx}$ after differentiation implicitly.

Important note : When differentiation implicitly, you must show that you are taking the derivative of both sides with respect to x.

Example 2:

Find the slope of the graph of $y^3 + y^2 - 5y - x^2 = -4$ at $(1,-3)$ by (a) finding $\frac{dy}{dx}$ then evaluating it at $(1,-3)$, then by (b) differentiation and plugging in $(1,-3)$ before solving for $\frac{dy}{dx}$.

Method ①:

$$3y^2y' + 2yy' - 5y' - 2x = 0$$
$$y'(3y^2 + 2y - 5) = 2x$$
$$y' = \frac{2x}{3y^2 + 2y - 5}$$
$$\left.\frac{dy}{dx}\right|_{(1,-3)} = \frac{2(1)}{3(-3)^2 + 2(-3) - 5}$$
$$= \frac{2}{27 - 6 - 5}$$
$$= \frac{2}{16}$$
$$= \frac{1}{8}$$

Method ②:

$$3y^2y' + 2yy' - 5y' - 2x = 0$$
$$3(-3)^2y' + 2(-3)y' - 5y' - 2(1) = 0$$
$$27y' - 6y' - 5y' = 2$$
$$16y' = 2$$
$$y' = \frac{2}{16}$$
$$y' = \frac{1}{8}$$

Example 3:

Find the coordinates where the graph $x^2 + y^2 = 25$ has horizontal and vertical tangent lines. At what point is the slope $\frac{3}{4}$?

$$x^2 + y^2 = 25 \quad \text{—— ①}$$
$$2x + 2yy' = 0$$
$$y' = \frac{-2x}{2y}$$
$$y' = \frac{-x}{y}$$

Horizontal tangent: $y' = 0$

$$0 = \frac{-x}{y}$$
$$x = 0$$

Sub in ①:

$$y^2 = 25$$
$$y = \pm 5$$

Vertical tangent: $y' = \infty$

$$-\frac{x}{y} = \frac{1}{0}$$
$$y = 0$$

Sub in ①:

$$x^2 = 25$$
$$x = \pm 5$$

$$m_t = \frac{3}{4}$$
$$\frac{3}{4} = \frac{-x}{y}$$
$$x = -\frac{3}{4}y \quad \text{—— ②}$$

Sub in ①:

$$\left(-\frac{3}{4}y\right)^2 + y^2 = 25$$
$$\frac{9}{16}y^2 + y^2 = 25$$
$$\frac{25}{16}y^2 = 25$$
$$y^2 = 16$$
$$y = \pm 4$$

Sub in ②:

At $y = 4$, $x = -3$

At $y = -4$, $x = 3$

$\therefore$ The graph has horizontal tangent lines at $(0,5)$ and $(0,-5)$ and vertical tangent lines at $(5,0)$ and $(-5,0)$

The graph has a slope of $\frac{3}{4}$ at $(3,-4)$ and $(-3,4)$

Example 4:

Find the coordinates (x, y) of any horizontal and vertical tangent lines to the curve given by the equation $2x^2 + xy + 4y^2 = 3$. Be sure to label your work.

$$4x + 1(y) + xy' + 8yy' = 0$$
$$y'(x+8y) = -4x - y$$
$$y' = \frac{-4x-y}{x+8y}$$

Horizontal tangent: $y' = 0$

$$0 = -4x - y$$
$$y = -4x$$
$$\because 2x^2 + xy + 4y^2 = 3 \quad — ①$$
$$\therefore 2x^2 + x(-4x) + 4(-4x)^2 = 3$$
$$2x^2 - 4x^2 + 64x^2 = 3$$
$$62x^2 = 3$$
$$x^2 = \frac{3}{62}$$
$$x = \pm\sqrt{\frac{3}{62}}$$
$$\because y = -4x$$

At $x = \sqrt{\frac{3}{62}}$, $y = \frac{-4\sqrt{3}}{\sqrt{62}}$ $\quad \therefore \left(\sqrt{\frac{3}{62}}, \frac{-4\sqrt{3}}{\sqrt{62}}\right)$

At $x = -\sqrt{\frac{3}{62}}$, $y = \frac{4\sqrt{3}}{\sqrt{62}}$ $\quad \left(-\sqrt{\frac{3}{62}}, \frac{4\sqrt{3}}{\sqrt{62}}\right)$

Vertical tangent : $y' = \infty$

$$\frac{1}{0} = \frac{-4x-y}{x+8y}$$
$$x + 8y = 0$$
$$x = -8y$$

Sub in ① :

$$2(-8y)^2 + (-8y)(y) + 4y^2 = 3$$
$$128y^2 - 8y^2 + 4y^2 = 3$$
$$124y^2 = 3$$
$$y^2 = \frac{3}{124}$$
$$y = \pm\sqrt{\frac{3}{124}}$$
$$\because x = -8y$$

$\therefore$ At $y = \sqrt{\frac{3}{124}}$, $x = \frac{-8\sqrt{3}}{\sqrt{124}}$ $\quad \left(\frac{-8\sqrt{3}}{\sqrt{124}}, \sqrt{\frac{3}{124}}\right)$

At $y = -\sqrt{\frac{3}{124}}$, $x = \frac{8\sqrt{3}}{\sqrt{124}}$ $\quad \left(\frac{8\sqrt{3}}{\sqrt{124}}, -\sqrt{\frac{3}{124}}\right)$

Example 5:

Determine the equation of the tangent line of $3\left(x^2 + y^2\right)^2 = 100xy$ at the point $(3,1)$

$$6(x^2+y^2)(2x+2yy') = 100(y+xy')$$
$$6(3^2+1^2)(2(3)+2(1)y') = 100(1+3y')$$
$$60(6+2y') = 100(1+3y') \Leftarrow \text{Divide by 20.}$$
$$3(6+2y') = 5(1+3y')$$
$$18 + 6y' = 5 + 15y'$$
$$13 = 9y'$$
$$\boxed{y' = \frac{13}{9}}$$

Equation of tangent line is :

$$y - 1 = \frac{13}{9}(x-3)$$
$$y = \frac{13}{9}x - \frac{13}{3} + 1$$
$$\boxed{y = \frac{13}{9}x - \frac{10}{3}}$$

Name____ Date________________________

Implicit Differentiation Practice

Show all work. No calculator unless otherwise stated.

1. Find $\frac{dy}{dx}$

(a) $x^3 - 3x^2y + 4xy^2 = 12$

$$\frac{d}{dx}\left[x^3 - 3x^2y + 4xy^2\right] = \frac{d}{dx}[12]$$

$$3x^2 - 6xy - 3x^2\frac{dy}{dx} + 4y^2 + 4x\left(2y\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}\left[-3x^2 + 8xy\right] = 6xy - 3x^2 - 4y^2$$

$$\boxed{\frac{dy}{dx} = \frac{6xy - 3x^2 - 4y^2}{8xy - 3x^2}}$$

or $\frac{dy}{dx} = \frac{3x^2 + 4y^2 - 6xy}{3x^2 - 8xy}$ *multiply by $\frac{-1}{-1}$

(b) $\sqrt{xy} = x + 3y$

$$\frac{d}{dx}\left[(xy)^{1/2}\right] = \frac{d}{dx}[x + 3y]$$

$$\frac{1}{2}(xy)^{-1/2}\left(1 \cdot y + x \cdot \frac{dy}{dx}\right) = 1 + 3\frac{dy}{dx}$$

$$\frac{1}{2}x^{-1/2}y^{-1/2}y^{1} + \frac{1}{2}x^{-1/2}x^{1}y^{-1/2} \cdot \frac{dy}{dx} = 1 + 3\frac{dy}{dx}$$

$$\frac{\sqrt{y}}{2\sqrt{x}} + \frac{\sqrt{x}}{2\sqrt{y}}\frac{dy}{dx} = 1 + 3\frac{dy}{dx}$$

$$\frac{\sqrt{y}}{2\sqrt{x}} - 1 = \frac{dy}{dx}\left[3 - \frac{\sqrt{x}}{2\sqrt{y}}\right]$$

$$\frac{dy}{dx} = \frac{\frac{\sqrt{y}}{2\sqrt{x}} - 1}{3 - \frac{\sqrt{x}}{2\sqrt{y}}}\left(\frac{2\sqrt{x}\sqrt{y}}{2\sqrt{x}\sqrt{y}}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{y - 2\sqrt{xy}}{6\sqrt{xy} - x}}$$

or $\frac{dy}{dx} = \frac{2\sqrt{xy} - y}{x - 6\sqrt{xy}}$ *multiply by $\frac{-1}{-1}$

c) $\left(y^2 + 2xy\right)^2 = 4(x+1)^2$

$$\frac{d}{dx}\left[\left(y^2 + 2xy\right)^2\right] = 4\frac{d}{dx}\left[(x+1)^2\right]$$

$$2\left(y^2 + 2xy\right)\left(2y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx}\right) = 8(x+1)$$

$$\left(y^2 + 2xy\right)\left(y\frac{dy}{dx} + y + x\frac{dy}{dx}\right) = 2(x+1)$$

$$y^3\frac{dy}{dx} + y^3 + y^2x\frac{dy}{dx} + 2xy^2\frac{dy}{dx} + 2xy^2 + 2x^2y\frac{dy}{dx} = 2x + 2$$

$$\frac{dy}{dx}\left(y^3 + 3y^2x + 2x^2y\right) = 2x + 2 - y^3 - 2xy^2$$

$$\frac{dy}{dx} = \frac{2x + 2 - y^3 - 2xy^2}{y^3 + 3y^2x + 2x^2y}$$

2. Find $\frac{dy}{dx}$ at the indicated point, then find the equation of both the tangent and normal lines.

(a) $y^2 = \frac{x^2-4}{x^2+4}$ at $(2,0)$

$\frac{d}{dx}[y^2] = \frac{d}{dx}\left[\frac{x^2-4}{x^2+4}\right]$

$2y\frac{dy}{dx} = \frac{(x^2+4)(2x)-(x^2-4)(2x)}{(x^2+4)^2}$

$\frac{dy}{dx} = \frac{2x^3+8x-2x^3+8x}{2y(x^2+4)^2}$

$\boxed{\frac{dy}{dx} = \frac{8x}{y(x^2+4)^2}}$

(b) $(x+y)^3 = x^3+y^3$ at $(-1,1)$

$\frac{d}{dx}[(x+y)^3] = \frac{d}{dx}[x^3+y^3]$

$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2+3y^2\frac{dy}{dx}$

$3(x+y)^2 + 3\frac{dy}{dx}(x+y)^2 = 3x^2+3y^2\frac{dy}{dx}$

$\frac{dy}{dx}\left[3(x+y)^2-3y^2\right] = 3x^2-3(x+y)^2$

$\boxed{\frac{dy}{dx} = \frac{x^2-(x+y)^2}{(x+y)^2-y^2}}$

$\left.\frac{dy}{dx}\right|_{(-1,1)} = \frac{1-0^2}{0^2-1} = -1 = m$

Tangent Line: $\boxed{y = 1-1(x+1)}$ Normal line: $\boxed{y = 1+1(x+1)}$

3. Find $\frac{d^2y}{dx^2}$ in terms of x and y.

(a) $x^2+y^2 = 36$

$\frac{d}{dx}[x^2+y^2] = \frac{d}{dx}[36]$

$2x+2y\frac{dy}{dx} = 0$

$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$

$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{-x}{y}\right]$

$\frac{d^2y}{dx^2} = \frac{(y)(-1)-(-x)\cdot}{y^2}$

$\frac{d^2y}{dx^2} = \frac{-y+x\left(-\frac{x}{y}\right)}{y^2}\left(\frac{y}{y}\right)$

$\boxed{\frac{dy}{dx} = \frac{-y^2-x^2}{y^3}}$

or $\frac{dy}{dx} = -\frac{x^2+y^2}{y^3}$

(b) $1-xy = x-y$

*Solve for y 1st!

$y-xy = x-1$

$y(1-x) = x-1$

$y = \frac{x-1}{-(x-1)}$

$y = -1,\ x \neq 1$

$\frac{dy}{dx} = 0,\ x \neq 1$

$\boxed{\frac{d^2y}{dx^2} = 0,\ x \neq 1}$

* simplify early & often

(c) $\sqrt[3]{x^2}+\sqrt[3]{y^2} = 1$

*Solve for y 1st!

$y^{2/3} = 1-x^{2/3}$

$y = (1-x^{2/3})^{3/2}$

$\frac{dy}{dx} = \frac{3}{2}(1-x^{2/3})^{1/2}\cdot\left(-\frac{2}{3}x^{-1/3}\right)$

$\frac{dy}{dx} = (-x^{-1/3})(1-x^{2/3})^{1/2}$

$\frac{d^2y}{dx^2} = \left(\frac{1}{3}x^{-4/3}\right)(1-x^{2/3})^{1/2} + (-x^{-1/3})\left(\frac{1}{2}(1-x^{2/3})^{-1/2}\cdot\left(-\frac{2}{3}x^{-1/3}\right)\right)$

$\frac{d^2y}{dx^2} = \frac{1}{3}x^{-4/3}(1-x^{2/3})^{1/2} + \frac{1}{3}x^{-2/3}(1-x^{2/3})^{-1/2}$

$\frac{d^2y}{dx^2} = \frac{1}{3}x^{-4/3}(1-x^{2/3})^{-1/2}\left[(1-\cancel{x^{2/3}})+\cancel{x^{2/3}}\right]$

factor out least powers!

$\boxed{\frac{d^2y}{dx^2} = \frac{1}{3\sqrt[3]{x^4}\sqrt{1-\sqrt[3]{x^2}}}}$

4. Determine the point(s) at which the graph of $y^4 = y^2 - x^2$ has either a horizontal or vertical tangent. Be sure to label which is which, if either exist.

$\frac{d}{dx}[y^4] = \frac{d}{dx}[y^2 - x^2]$

$4y^3 \cdot \frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$

$\frac{dy}{dx}[4y^3 - 2y] = -2x$

$\frac{dy}{dx} = \frac{-2x}{4y^3 - 2y}$

$\frac{dy}{dx} = \frac{2x}{2y - 4y^3}$

Horz. tangents when

$\frac{dy}{dx} = \frac{0}{\neq 0}$

$2x = 0$

$\boxed{x = 0}$

when $x=0$:

$y^4 = y^2 - 0^2$

$y^4 - y^2 = 0$

$y^2(y^2 - 1) = 0$

$y = 0, y = -1, y = 1$

$\boxed{\text{pts: } (0,0), (0,-1), (0,1)}$

Vert. tangents when

$\frac{dy}{dx} = \frac{\neq 0}{0}$

$2y - 4y^3 = 0$

$2y(1 - 2y^2) = 0$

$y = 0, \quad y = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

when $y=0$: $0^4 = 0^2 - x^2$

$x = 0$

when $y = \pm\frac{\sqrt{2}}{2}$:

$\left(\pm\frac{\sqrt{2}}{2}\right)^4 = \left(\pm\frac{\sqrt{2}}{2}\right)^2 - x^2$

$\frac{4}{16} = \frac{2}{4} - x^2$

$\frac{1}{4} - \frac{1}{2} = -x^2$

$x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

$\boxed{\text{pts: } (0,0), \left(\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right), \left(-\frac{1}{2}, \pm\frac{\sqrt{2}}{2}\right)}$

5. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

when curve crosses x-axis, $y=0$:

$x^2 + 0 + 0 = 7$

$x = -\sqrt{7}$ or $x = \sqrt{7}$

when $x = \sqrt{7}$:

$\cancel{7} + \sqrt{7}y + y^2 = \cancel{7}$

$y(\sqrt{7} + y) = 0$

$y = 0$ or $y = -\sqrt{7}$

$\boxed{\text{pts: } (\sqrt{7}, 0), (\sqrt{7}, -\sqrt{7})}$

when $x = -\sqrt{7}$:

$\cancel{7} - \sqrt{7}y + y^2 = \cancel{7}$

$y(y - \sqrt{7}) = 0$

$y = 0, y = \sqrt{7}$

$\boxed{\text{pts: } (-\sqrt{7}, 0), (-\sqrt{7}, \sqrt{7})}$

So the TWO x-intercepts of the curve are $\boxed{(-\sqrt{7}, 0) \text{ and } (\sqrt{7}, 0)}$

(as shown in the graph at right)

parallel tangent lines have the same slope.

$\frac{d}{dx}[x^2 + xy + y^2] = \frac{d}{dx}[7]$

$2x + (1)(y) + (x)\left(\frac{dy}{dx}\right) + (2y)\left(\frac{dy}{dx}\right) = 0$

$\frac{dy}{dx}(x + 2y) = -2x - y$

$\boxed{\frac{dy}{dx} = -\frac{2x + y}{x + 2y}}$

$\left.\frac{dy}{dx}\right|_{(-\sqrt{7},0)} = -\frac{-2\sqrt{7} + 0}{-\sqrt{7} + 2(0)} = \boxed{-2}$

$\left.\frac{dy}{dx}\right|_{(\sqrt{7},0)} = -\frac{2\sqrt{7} + 0}{\sqrt{7} + 2(0)} = \boxed{-2}$

So the common slope is -2 at the x-intercepts.

6. Find the equations of the normal lines to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$

$2x + y = 0$

$y = -2x$

slope $= -2$

if the normal lines are to be parallel the given line, their slopes must equal -2, so the tangent lines at those points must be $\boxed{\frac{1}{2}}$

$\frac{d}{dx}[xy + 2x - y] = \frac{d}{dx}[0]$

$(1)(y) + (x)\left(\frac{dy}{dx}\right) + 2 - \frac{dy}{dx} = 0$

$\frac{dy}{dx}(x - 1) = -y - 2$

$\frac{dy}{dx} = \frac{-y-2}{x-1}$

$\boxed{\frac{dy}{dx} = \frac{y+2}{1-x}}$

$\frac{dy}{dx} = \frac{1}{2}$

$\frac{y+2}{1-x} = \frac{1}{2}$

$2y + 4 = 1 - x$

$x = 1 - 4 - 2y$

$\boxed{x = -3 - 2y}$

When $x = -3 - 2y$:

$(-3-2y)y + 2(-3-2y) - y = 0$

$-3y - 2y^2 - 6 - 4y - y = 0$

$-2y^2 - 8y - 6 = 0$

$-2(y^2 + 4y + 3) = 0$

$-2(y+1)(y+3) = 0$

$\boxed{y = -1 \text{ or } y = -3}$

when $y = -1$:

$-x + 2x + 1 = 0$

$\boxed{x = -1}$

pt: $(-1, -1)$

Normal line eq:

$m = -2$, pt $(-1, -1)$

$\boxed{y = -1 - 2(x+1)}$

when $y = -3$:

$-3x + 2x + 3 = 0$

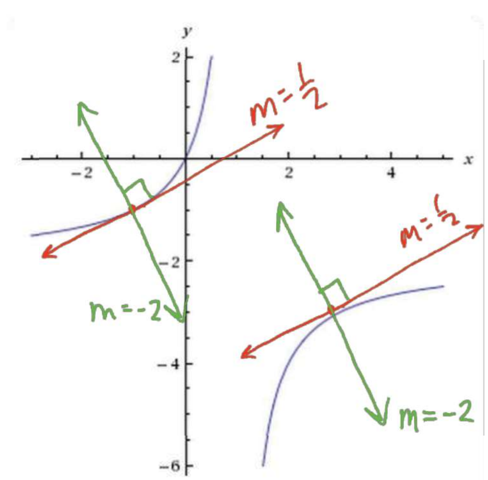
$\boxed{x = 3}$

pt: $(3, -3)$

Normal Line Eq:

$m = -2$, pt $(3, -3)$

$\boxed{y = -3 - 2(x-3)}$



7. The slope of the tangent is -1 at the point $(0,1)$ on $x^3 - 6xy - ky^3 = a$, where k and a are constants. The values of the constants a and k are what?

At $(0,1)$:

$0^3 - 6(0)(1) - k(1^3) = a$

$\boxed{-k = a}$

$\frac{d}{dx}\left[x^3 - 6xy - ky^3\right] = \frac{d}{dx}[a]$

$3x^2 - 6y - 6x\frac{dy}{dx} - 3ky^2\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(-6x - 3ky^2\right) = 6y - 3x^2$

$\frac{dy}{dx} = \frac{6y - 3x^2}{-6x - 3ky^2}$

$\boxed{\frac{dy}{dx} = \frac{3x^2 - 6y}{6x + 3ky^2}}$

$\left.\frac{dy}{dx}\right|_{(0,1)} = -1 = \frac{3(0^2) - 6(1)}{6(0) + 3k(1^2)}$

$-1 = \frac{-6}{3k}$

$3k = 6$

$\boxed{k = 2}$

$-k = a$

$a = -k$

Multiple Choice

B 8. Find y' when $xy+5x+2x^2=4$.

(A) $y'=\frac{5+2x-y}{x}$ (B) $y'=-\frac{y+5+4x}{x}$ (C) $y'=-(y+5+4x)$ (D) $y'=\frac{y+5+2x}{x}$

(E) $y'=-\frac{y+5+2x}{x}$ (F) $y'=\frac{y+5+4x}{x}$

A 9. Find $\frac{dy}{dx}$ when $\frac{3}{\sqrt{x}}+\frac{2}{\sqrt{y}}=4$ (A) $\frac{dy}{dx}=-\frac{3}{2}\left(\frac{y}{x}\right)^{3/2}$ (B) $\frac{dy}{dx}=\frac{3}{2}(xy)^{1/2}$

(C) $\frac{dy}{dx}=-\frac{2}{3}\left(\frac{x}{y}\right)^{3/2}$ (D) $\frac{dy}{dx}=\frac{2}{3}\left(\frac{x}{y}\right)^{3/2}$ (E) $\frac{dy}{dx}=\frac{3}{2}\left(\frac{y}{x}\right)^{3/2}$ (F) $\frac{dy}{dx}=\frac{2}{3}(xy)^{1/2}$

D 10. Find the equation of the tangent line to the graph of $y^2-xy-12=0$ at the point $(1,4)$.

(A) $3y=2x+10$ (B) $3y+2x=10$ (C) $y=4x$ (D) $7y=4x+24$ (E) $7y+4x=24$

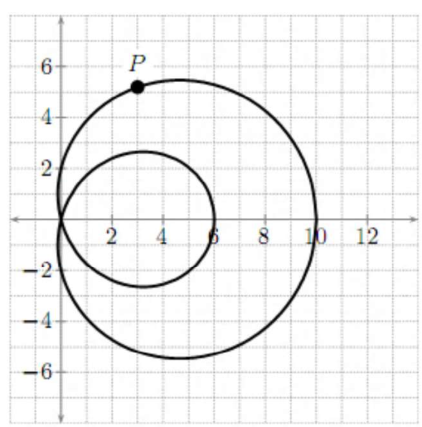
E 11. The slope of the tangent line to the graph of $x^3-2y^3+xy=0$ at the point $(1,1)$ is

(A) $-\frac{4}{5}$ (B) $\frac{3}{2}$ (C) $-\frac{5}{4}$ (D) $\frac{5}{4}$ (E) $\frac{4}{5}$ (F) $-\frac{2}{3}$

D 12. Determine $\frac{d^2y}{dx^2}$ when $4x^2+3y^2=4$

(A) $\frac{d^2y}{dx^2}=\frac{16}{9y^2}$ (B) $\frac{d^2y}{dx^2}=-\frac{16}{9y^2}$ (C) $\frac{d^2y}{dx^2}=-\frac{4}{9y^3}$ (D) $\frac{d^2y}{dx^2}=-\frac{16}{9y^3}$ (E) $\frac{d^2y}{dx^2}=\frac{16}{9y^3}$

D 13. The graph of the equation $\left(x^2+y^2-8x\right)^2=4\left(x^2+y^2\right)$ is shown at right. Find the equation of the tangent line to the graph at the point $(3,3\sqrt{3})$.



(A) $y=\frac{1}{\sqrt{3}}\left(\frac{5}{3}x-12\right)$ (B) $y=\frac{1}{\sqrt{3}}\left(\frac{5}{3}x+12\right)$ (C) $y=\frac{1}{\sqrt{3}}\left(\frac{3}{5}x+\frac{6\sqrt{3}}{5}\right)$

(D) $y=\frac{1}{\sqrt{3}}\left(\frac{3}{5}x-\frac{36}{5}\right)$ (E) $y=\frac{1}{\sqrt{3}}\left(\frac{3}{5}x+\frac{6}{5}\right)$

Related Rate of Change

Introduction

Change is an essential feature of the real world. In many situations a change in one quantity causes a change in another quantity or occurs together with a change in another quantity, with the result that the two rates of change are related. Consider the following problem:

> *An oil spill from a tanker spreads out in a circular pattern, centred at the tanker's position. If the edge of the oil spill is moving outwards at 2 m/s, find the rate of increase of the contaminated area when the radius is 500 m.*

This is an example of a "related-rates" problem. We are given the rate of change of radius with respect to time, $\frac{dr}{dt}$, and wish to calculate the rate of change of area with respect to time, $\frac{dA}{dt}$. Since area and radius are related by $A = \pi r^2$, and since both depend on time t, we can calculate $\frac{dA}{dt}$ by differentiating both sides of the equation $A = \pi r^2$ with respect to t. Differentiating both sides of an equation that relates two or more time-dependent variables forms the basis for the method of solving related-rates problems.

Strategy for Solving Related Rates Problems

1. **Read the problem carefully.**
2. **Draw and label a diagram if possible.**
3. **Introduce notation. Assign symbols to all quantities.**
4. **Express the given information and required rate in terms of derivatives.**
5. **Write an equation that relates the various quantities of the problem. You may need to know a formula for volume, area, surface area or Pythagoras.**
6. **Differentiate both sides of the equation with respect to time (t). You will probably use the chain rule.**
7. **Substitute the given information in the resulting equation and solve for the unknown rate.**

RESIST TEMPTATION

Do not substitute numerical information until the end of the problem!

BALLOON

Air is been pumped into a spherical balloon so that its volume increases at a rate of 80 cm^3/sec. How fast is the radius increasing when the diameter is 60 cm?

Relevant formula:	$V = \frac{4}{3}\pi r^3$ Let V be the volume, r be the radius and t be the time.
Known quantities:	$\frac{dV}{dt} = 80 \text{cm}^3/\text{s}$
Unknown quantity:	$\frac{dr}{dt} = ?$ when $r = 30$ cm
Differentiate wrt "t"	$\frac{dV}{dt} = \frac{4}{3}(3)\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$
Substitute $\frac{dV}{dt} = 80$; $r = 30$ →	$(80) = 4\pi(30)^2 \frac{dr}{dt}$ $\frac{dr}{dt} = \frac{80}{3600\pi}$ → $\frac{dr}{dt} = \frac{1}{45\pi}$ ∴ The radius is increasing at $\frac{1}{45\pi}$ cm/s. Answer: $\frac{1}{45\pi}$ cm/s

SNOWBALL If a snowball melts so that its surface area decreases at a rate of 1 cm^2/sec, find the rate at which the diameter decreases when the diameter is 12 cm.	
Relevant formulae:	$S = 4\pi r^2$ $\because D = 2r$ $\therefore S = 4\pi\left(\frac{D}{2}\right)^2$ $\boxed{S = \pi D^2}$ Let S be the surface area, D be the diameter and t be the time.
Known quantities:	$\frac{dS}{dt} = -1\ cm^2/s$
Unknown quantity:	$\frac{dD}{dt} = ?$ when $D = 12\ cm$
Differentiate wrt "t"	$\frac{dS}{dt} = 2\pi D \frac{dD}{dt}$
substitute $\frac{dS}{dt} = -1$; $D = 12$ $\rightarrow$	$(-1) = 2\pi(12)\frac{dD}{dt}$ $\frac{dD}{dt} = \frac{-1}{24\pi}$ $\therefore$ The diameter is decreasing at $\frac{1}{24\pi}$ cm/s. **Answer**: $\frac{-1}{24\pi}$ cm/s

CIRCLE PROBLEM

A golf ball chipped into a water hazard creates a circular ripple effect. If the radius of the ripple is increasing at 0.8 m/s, how fast is the area changing when the radius is 6m?

Let r be the radius, A be the area and t be the time.

Given: $\frac{dr}{dt} = 0.8\ m/s$

Find: $\frac{dA}{dt} = ?$ when $r = 6\ m$

Formula: $A = \pi r^2$

Differentiate: $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

Substitute: $\frac{dA}{dt} = 2\pi(6)(0.8)$

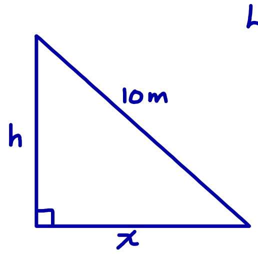
$= 9.6\pi$

$\therefore$ The area is changing at $9.6\pi\ m^2/s$.

Answer: $9.6\pi \frac{m^2}{s}$

THE FALLING LADDER

A ladder is sliding down along a vertical wall. If the ladder is 10 meters long and the top is slipping at the constant rate of 10 m/s, how fast is the bottom of the ladder moving along the ground when the bottom is 6 meters from the wall?



Let the diagram be as labelled.

Given: $\frac{dh}{dt} = -10\ m/s$

Find: $\frac{dx}{dt} = ?$ when $x = 6m$

$h^2 = 10^2 - 6^2$

$h^2 = 64$

$h = 8, h > 0$

Formula: $x^2 + h^2 = 10^2$

Differentiate: $2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$

Sub: $2(6)\frac{dx}{dt} + 2(8)(-10) = 0$

$\frac{dx}{dt} = \frac{80}{6}$

$\frac{dx}{dt} = \frac{40}{3}$

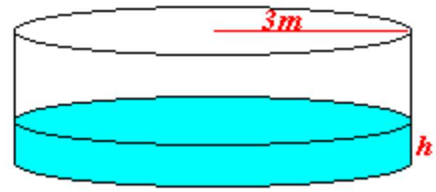
$\therefore$ The bottom of the ladder is moving at $\frac{40}{3}$ m/s.

Answer: $13\frac{1}{3}$ m/s

CYLINDER ROBLEM

A hose is filling a cylindrical swimming pool of radius 3m at a rate of 50 L / min. At what rate is the water level rising?

(Recall: 1 L / min = 0.001 m³ / min)



Let V be the volume, h be the height, r be the radius and t be time.

Given : $\frac{dV}{dt} = 0.05\ m^3/min$

Find : $\frac{dh}{dt} = ?$

Formula: $V = \pi r^2 h$

Sub in r=3 since it is a constant:

$V = \pi(3)^2 h$

$V = 9\pi h$

Differentiate: $\frac{dV}{dt} = 9\pi \frac{dh}{dt}$

Substitute: $(0.05) = 9\pi \cdot \frac{dh}{dt}$

$\frac{0.05}{9\pi} = \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{5}{900\pi}$

$\frac{dh}{dt} = \frac{1}{180\pi}$

∴ The water level is rising at $\frac{1}{180\pi}$ m/min.

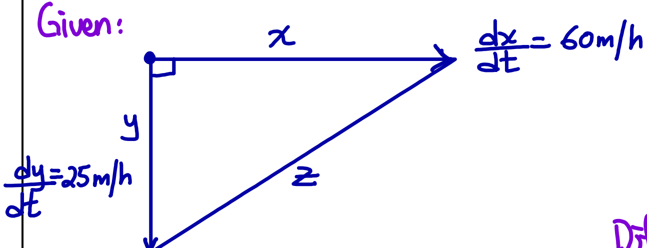
Answer: $\frac{1}{180\pi}$ m/min

CAR PROBLEM #1

Two cars start moving from the same point. Car A is traveling east at 60 m/h and car B is traveling south at 25 m/h. At what rate is the distance between them increasing 4 hours later?

Let z be the distance at any instant between the 2 cars.

Given:



Find: $\frac{dz}{dt} = ?$ when $t=4$

$x = 60\frac{m}{h} \times 4h = 240m$

$y = 25\frac{m}{h} \times 4h = 100m$

$z = \sqrt{240^2 + 100^2}$

$z = \sqrt{67600}$

$z = 260$

Formula: $z^2 = x^2 + y^2$

Differentiate: $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

Substitute: $2(260)\frac{dz}{dt} = 2(240)(60) + 2(100)(25)$

$\frac{dz}{dt} = \frac{16900}{260}$

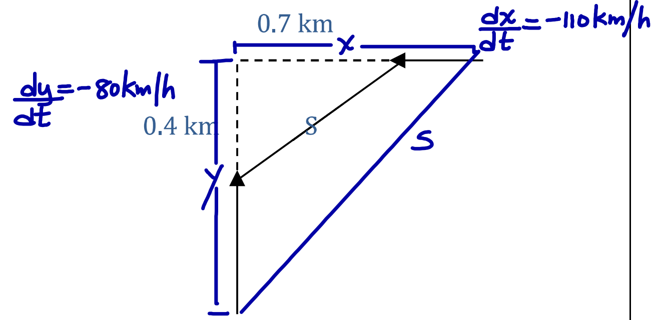
$\frac{dz}{dt} = 65$

∴ The distance between the 2 cars is increasing at 65 m/h.

Answer: 65 m/h

CAR PROBLEM #2

Car A is traveling north at 80 km/h and car B is traveling west at 110 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.4 km and car B is 0.7 km from the intersection?



Let s be the distance between the 2 cars at any instant.

Given: $\frac{dy}{dt} = -80\text{ km/h}$; $\frac{dx}{dt} = -110\text{ km/h}$

Find: $\frac{ds}{dt} = ?$ when $x = 0.7$ km and $y = 0.4$ km

Formula: $x^2 + y^2 = s^2$

Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$

Substitute: $2(0.7)(-110) + 2(0.4)(-80) = 2(\sqrt{0.65})\frac{ds}{dt}$

$\frac{ds}{dt} = \frac{-77-32}{\sqrt{0.65}}$

$\frac{ds}{dt} \doteq -135$

$s = \sqrt{0.7^2 + 0.4^2}$

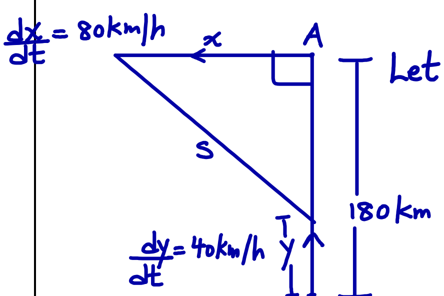
$s = \sqrt{0.65}$

Answer: −135.2 km/h

$\therefore$ The cars are approaching at 135 km/h

CAR PROBLEM #3

A train is 180km south of town **A**, travelling north at 40km/h. A car leaves town **A** driving west at 80km/h. At what rate is the distance between them changing 2 hours later?



Let s be the distance between the train and the car at any instant

Formula: $s^2 = x^2 + (180-y)^2$

Differentiate: $\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}(180-y)\left(-\frac{dy}{dt}\right)$

$x = 80\frac{\text{km}}{\text{h}} \times 2\text{h} = 160\text{ km}$

$y = 40\frac{\text{km}}{\text{h}} \times 2\text{h} = 80\text{ km}$

$s = \sqrt{160^2 + (180-80)^2}$

$s = \sqrt{35600}$

$s = 20\sqrt{89}$

Substitute: $(20\sqrt{89})\frac{ds}{dt} = (160)(80) + (100)(-40)$

$\frac{ds}{dt} = \frac{12800 - 4000}{20\sqrt{89}}$

$\frac{ds}{dt} \doteq 46.6$

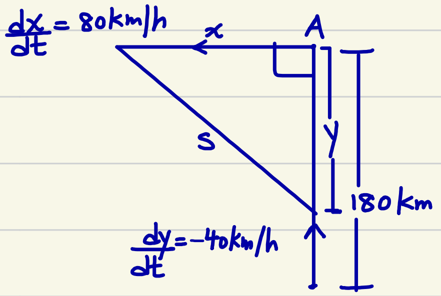
$\therefore$ The distance between the train and the car is increasing at 46.6 km/h.

Note: $\frac{dy}{dt}$ is not negative since y is increasing with time.

Car Problem #3 Method ②:

A train is 180km south of town **A**, travelling north at 40km/h. A car leaves town **A** driving west at 80km/h. At what rate is the distance between them changing 2 hours later?

Let s be the distance between the train and the car at any instant



$x = 80\frac{km}{\cancel{h}} \times 2\cancel{h} = 160$ km

$y = 180\text{km} - \left(40\frac{km}{\cancel{h}} \times 2\cancel{h}\right)$
$= 100$ km

$S = \sqrt{160^2 + 100^2}$

$S = \sqrt{35600}$

$S = 20\sqrt{89}$

$$s^2 = x^2 + y^2$$

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$$

$$(20\sqrt{89})\frac{ds}{dt} = (160)(80) + (100)(-40)$$

$$\frac{ds}{dt} = \frac{12800 - 4000}{20\sqrt{89}}$$

$$\frac{ds}{dt} \doteq 46.6$$

∴ The distance between the train and the car is changing at 46.6 km/h.

Note: $\frac{dy}{dt}$ is negative since distance is decreasing with time.

SHIP PROBLEM

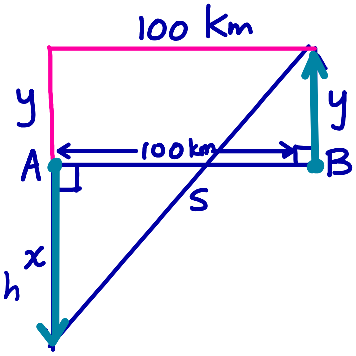
At noon, ship A is 100 km west of ship B. Ship A is sailing south at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between ships changing at 4:00pm?

Let s be the distance between the 2 ships at any instant

Given:

$\frac{dy}{dt} = 25 \text{ km/h}$

$\frac{dx}{dt} = 35 \text{ km/h}$



Formula:

$$s^2 = (x+y)^2 + 100^2$$

Differentiate:

$$\cancel{2}s\frac{ds}{dt} = \cancel{2}(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

Find: $\frac{ds}{dt} = ?$ at 4:00pm

$x = 35\frac{\text{km}}{\text{h}} \times 4 \text{ h}$

$= 140 \text{ km}$

$y = 25\frac{\text{km}}{\text{h}} \times 4 \text{ h}$

$= 100 \text{ km}$

$s^2 = (100+140)^2 + 100^2$

$s^2 = 67600$

$s = 260 \text{ km}$

Substitute:

$$(260)\frac{ds}{dt} = [(140)+(100)][(35)+(25)]$$

$$\frac{ds}{dt} = \frac{(240)(60)}{260}$$

$$\frac{ds}{dt} \doteq 55.4$$

$\therefore$ The distance between the ships is increasing at 55.4 km/h.

Answer: 55.4 km/h

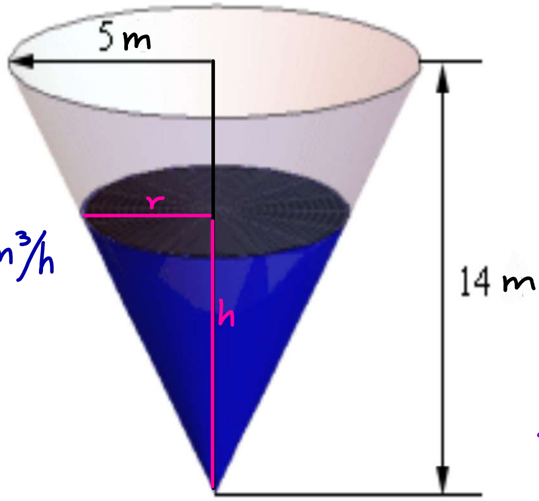
CONE PROBLEM #1

A tank of water in the shape of a cone is leaking water at a constant rate of $2\text{m}^3/\text{h}$. The base radius of the tank is 5 m and the height of the tank is 14 m.

(a) At what rate is the depth of the water in the tank changing when the depth of the water is 6 m?

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 m?

Diagram:



Given:

$$\frac{dV}{dt} = -2\ \text{m}^3/\text{h}$$

$$V = \frac{1}{3}\pi r^2 h$$

Formula: Using similar triangles,

$$\frac{r}{5} = \frac{h}{14}$$

$$r = \frac{5}{14}h$$

$$V = \frac{\pi}{3}\left(\frac{5}{14}h\right)^2 h$$

$$V = \frac{25\pi h^3}{588}$$

Differentiate:

$$\frac{dV}{dt} = \frac{25\pi}{588}\left(3h^2\frac{dh}{dt}\right)$$

Substitute:

$$(-2) = \frac{25\pi(6)^2}{196}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{-2(196)}{25\pi(36)}$$

$$\frac{dh}{dt} = \frac{-98}{225\pi}$$

a) Find: $\frac{dh}{dt} = ?$ when $h = 6\text{m}$

$\therefore$ The depth of the water is decreasing at $\frac{98}{225\pi}$ m/h.

b) Find: $\frac{dr}{dt} = ?$ when $h = 6\text{m}$

Formula:

$$\frac{r}{5} = \frac{h}{14}$$

$$\therefore h = \frac{14}{5}r$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{\pi}{3}r^2\left(\frac{14r}{5}\right)$$

$$V = \frac{14\pi r^3}{15}$$

Differentiate:

$$\frac{dV}{dt} = \frac{14\pi}{15}\left(3r^2\frac{dr}{dt}\right)$$

Substitute:

$$-2 = \frac{14\pi}{5}\left(\frac{15}{7}\right)^2\frac{dr}{dt}$$

$$\because 6 = \frac{14r}{5}$$

$$\therefore r = \frac{15}{7}$$

$$\frac{dr}{dt} = \frac{-2(5)(49)}{14\pi(15)(15)}$$

$$\frac{dr}{dt} = \frac{-7}{45\pi}$$

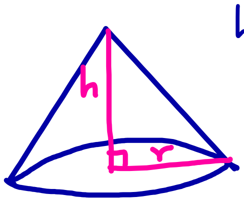
$\therefore$ The radius of the top of the water is decreasing at $\frac{7}{45\pi}$ m/h.

Answer: a) $\frac{-98}{225\pi}$ m/h b) $\frac{-7}{45\pi}$ m/h

CONE PROBLEM #2

Gravel is being dumped from a conveyor belt at a rate of 25 cubic feet per minute and it forms a pile in the shape of a cone whose base diameter is twice the height. How fast is the height increasing when the pile is 12 feet high?

Let V represent the volume, r be the radius, h be the height and t be time.



Formula: $V = \frac{1}{3}\pi r^2 h$

$\because D = 2h$
$2r = 2h$
$\therefore h = r$

$\therefore V = \frac{\pi}{3}(h^2)(h)$

$V = \frac{\pi h^3}{3}$

Given: $\frac{dV}{dt} = 25\ ft^3/min$

Find: $\frac{dh}{dt} = ?$ when $h = 12\ ft$

Differentiate: $\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$

Substitute: $25 = \pi(12)^2 \frac{dh}{dt}$

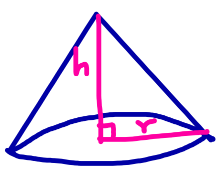
$\frac{dh}{dt} = \frac{25}{144\pi}$

$\therefore$ The height is increasing at $\frac{25}{144\pi}$ ft/min.

CONE PROBLEM #3

Sand pouring from a conveyor belt forms a conical pile, the radius of which is $\frac{3}{4}$ of the height. If the sand is piling up at a constant rate of $\frac{1}{2}$ m^3/min, at what rate is the height of the pile growing 3 min after the pouring starts.

Let V be the volume of the cone, h be the height, r be the radius and t be time.



Formula: $r = \frac{3}{4}h \rightarrow V = \frac{\pi r^2 h}{3}$

$V = \frac{\pi}{3}\left(\frac{3}{4}h\right)^2 h$

$V = \frac{3\pi h^3}{16}$

Given: $\frac{dV}{dt} = \frac{1}{2}\ m^3/min$

Find: $\frac{dh}{dt} = ?$ when $t = 3$ min

After 3 min, $V = \frac{1}{2}\ m^3/min \times 3\ min$

$V = \frac{3}{2}\ m^3/min$

$\because V = \frac{3\pi}{16}h^3$

$\therefore \frac{3}{2} = \frac{3\pi}{16}h^3$

$\frac{8}{\pi} = h^3$

$\left(\frac{8}{\pi}\right)^{\frac{1}{3}} = h$

Differentiate: $\frac{dV}{dt} = \frac{3\pi}{16}\left(3h^2\frac{dh}{dt}\right)$

Substitute: $\frac{1}{2} = \frac{9\pi}{16}\left(\frac{8}{\pi}\right)^{\frac{2}{3}}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{16(\pi)^{\frac{2}{3}}}{18\pi(8)^{\frac{2}{3}}}$

$\frac{dh}{dt} \doteq 0.152$

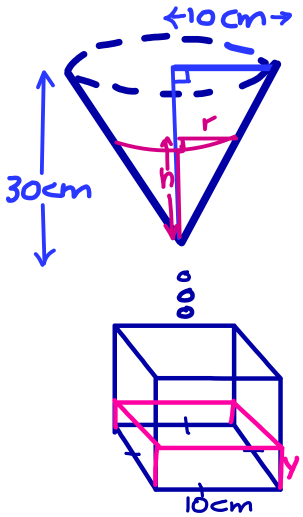
$\therefore$ The height is growing at 0.152 m/min.

Answer. 0.152 m/min

CONE PROBLEM #4

A cone with height of 30cm and a diameter of 20cm contains water. Water is leaking out of the conical cup at a constant rate such that the depth of the water, ***h***, is decreasing at 2 cm/min when the depth of the water is 20 cm. The water is then collecting in a box with a square base 10 cm per side. At what rate is the depth of the water, ***y***, increasing in the box?

Let V be the volume of the cone, h be the height, r be the radius and t be the time.



$\frac{dh}{dt} = -2\text{cm/min}$ when $h = 20\text{cm}$

Formula for the cone:

Similar △s:

$$\frac{r}{10} = \frac{h}{30}$$

$$r = \frac{h}{3}$$

$$V_1 = \frac{1}{3}\pi r^2 h$$

$$V_1 = \frac{\pi}{3}\left(\frac{h}{3}\right)^2 h$$

$$V_1 = \frac{\pi h^3}{27}$$

Differentiate:

$$\frac{dV_1}{dt} = \frac{\pi}{27}\left(3h^2\frac{dh}{dt}\right)$$

$$\frac{dV_1}{dt} = \frac{\pi(20)^2}{9}(-2)$$

$$\frac{dV_1}{dt} = \frac{-800\pi}{9}$$

Note: $\frac{dV_1}{dt}$ is negative since the water is leaking out of the cone.

Formula for the box:

$$V_2 = 100y$$

Differentiate:

$$\frac{dV_2}{dt} = 100\frac{dy}{dt}$$

Change in volume in the cone is equal to the change in volume in the box (i.e. no water loss).

$$\therefore \frac{dV_1}{dt} = \frac{dV_2}{dt}$$

$$\therefore \frac{800\pi}{9} = 100\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{800\pi}{9(100)}$$

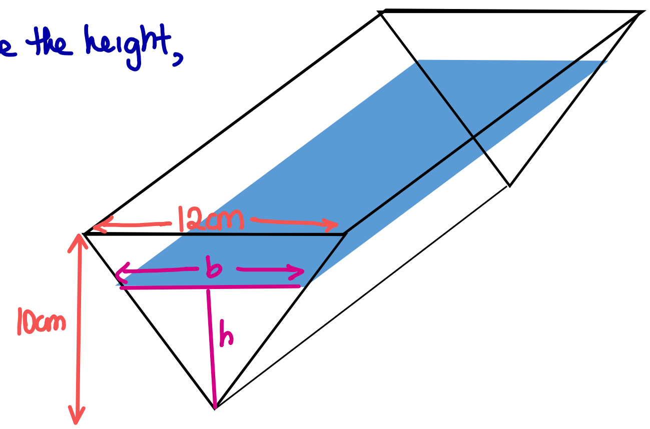
$$\frac{dy}{dt} = \frac{8\pi}{9}$$

$\therefore$ The depth of the water in the box is increasing at $\frac{8\pi}{9}$ cm/min.

Note: $\frac{dV_2}{dt}$ is positive since the box is being filled with water.

TROUGH PROBLEM

Water pours into a triangular trough at a rate of 0.5 L/min. The trough has a height of 10 cm, a width of 12 cm and a length of 8 m. How fast is the depth of the water changing when the depth is 8 cm?



Let b be the base of the triangle face, h be the height, V be the volume and t be time.

Given: $\frac{dV}{dt} = 0.5 \text{ L/min}$
$= 500 \text{ cm}^3\text{/min}$ ← convert units
$1000\text{cm}^3 = 1 \text{ L}$

Need to find: $\frac{dh}{dt} = ?$ when $h = 8$ cm

Formula: $V = (\text{Area of base})(\text{length of prism})$
$= \frac{1}{2} b h \ell$
$\because$ length of trough is a constant

$\therefore V = \frac{1}{2} b h (800)$
$= 400 bh$

Use similar triangles to reduce variables:

$\frac{b}{12} = \frac{h}{10}$
$b = \frac{3}{5} h$

$\therefore V = 800\left(\frac{3}{5}h\right)h$
$= 480 h^2$

Differentiate: $\frac{dV}{dt} = 480\left(2h \frac{dh}{dt}\right)$
$= 960 h \frac{dh}{dt}$

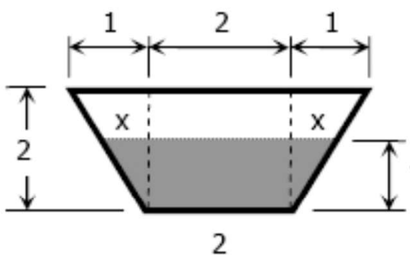
Substitute: $(500) = 960(8)\frac{dh}{dt}$
$\frac{dh}{dt} = 0.065$

$\therefore$ The depth of the water is rising at a rate of 0.065 cm/min.

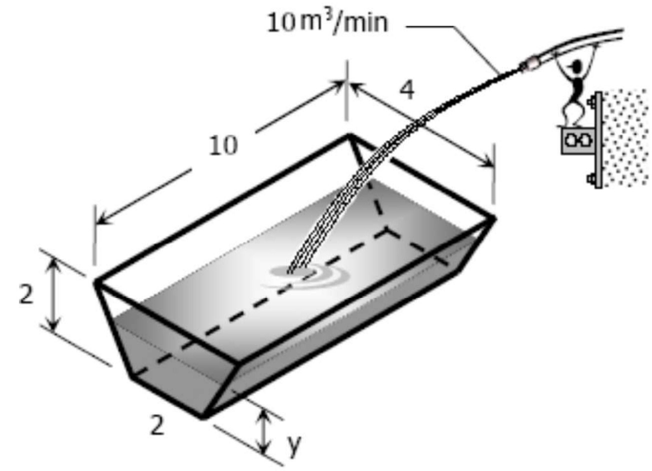
1. Water is being pumped into a trough that is 4.5m long and has a cross section in the shape of an equilateral triangle 1.5m on a side. If the rate of inflow is 2 cubic meters per minute how fast is the water level rising when the water is 0.5m deep?

2. The cross section of a 10-meter trough is an isosceles trapezoid with a 2-meter lower base, a 4-meter upper base, and an altitude of 2 meters.

a) Write an expression for the volume of water in the trough as a function of y.



Cross Section



b) Water is running into the trough at a rate of 10 cubic meters per minute. How fast is the water level rising when the water is 0.5 meter deep?

3. A trough filled with water is 2 m long and has a cross section in the shape of an isosceles trapezoid 30 cm wide at the bottom, 60 cm wide at the top, and a height of 50 cm., if the trough leaks water at the rate of 2000 cm^3/min, how fast is the water level falling when the water is 20 cm deep?

4. A trough is 12 feet long and 3 feet across the top. Its ends are isosceles triangles with altitudes of 3 feet

   a) If water is being pumped into the trough at a rate of 2 cubic feet per minute, how fast is the water level rising when the depth is 1 foot?

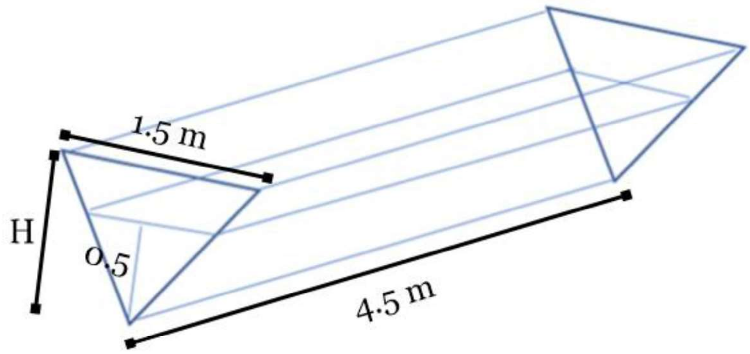
   b) If the water is rising at a rate of 3/8 inch per minute when h = 2, determine the rate at which the water is being pumped into the trough

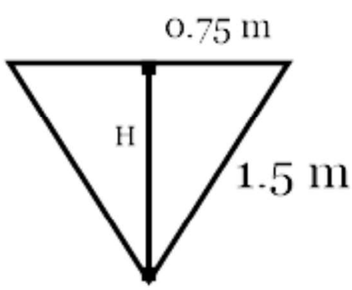
SOLUTIONS

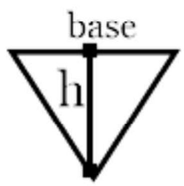
1. Water is being pumped into a trough that is 4.5m long and has a cross section in the shape of an equilateral triangle 1.5m on a side. If the rate of inflow is 2 cubic meters per minute how fast is the water level rising when the water is 0.5m deep?

$$\frac{dv}{dt} = 2m^3 / min$$

$$\frac{dh}{dt} = ? \text{ when } h = 0.5m$$







$$\frac{1.5}{\frac{3}{4}\sqrt{3}} = \frac{base}{h}$$

$$base = \frac{2h\sqrt{3}}{3}$$

$$H = \sqrt{\frac{9}{4} - \frac{9}{16}}$$

$$= \frac{3}{4}\sqrt{3}$$

$$V = (\text{area of base of trough})(\text{length of trough})$$

$$= (\text{area of the } \Delta \text{ of water})(4.5m)$$

$$= \frac{1}{2} \cdot \text{base of the } \Delta \text{ of water} \cdot \text{height of the } \Delta \text{ of water.}(4.5m)$$

$$= \frac{1}{2}\left(\frac{2h\sqrt{3}}{3}\right)(h).(4.5m)$$

$$= \frac{3\sqrt{3}}{2}h^2$$

$$V = \frac{3\sqrt{3}}{2}h^2$$

$$\frac{dv}{dt} = 3\sqrt{3}\,h\frac{dh}{dt}$$

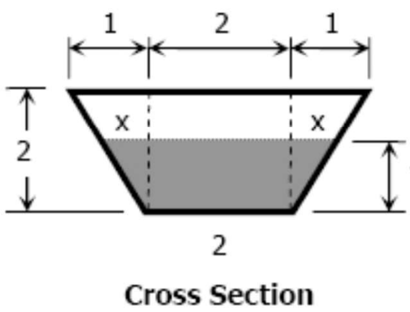
$$2 = 3\sqrt{3}\left(\frac{1}{2}\right)\frac{dh}{dt}$$

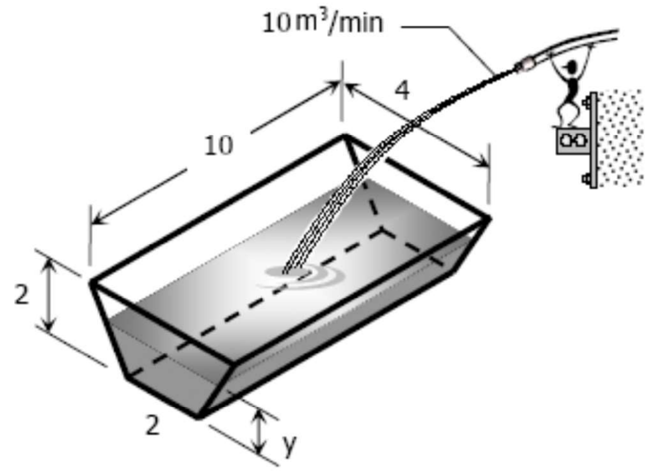
$$\frac{dh}{dt} = \frac{4\sqrt{3}}{9} \text{ m / min}$$

SOLUTIONS

2. The cross section of a 10-meter trough is an isosceles trapezoid with a 2-meter lower base, a 4-meter upper base, and an altitude of 2 meters.

a) Write an expression for the volume of water in the trough as a function of y.





$V = \frac{1}{2}(a+b)h \cdot l$

$$V = \frac{1}{2}\left[2 + (2 + 2x)\right]y(10)$$

$$V = 20y + 10xy$$

From similar triangles: $\frac{x}{y} = \frac{1}{2}$

$$x = \frac{1}{2}y$$

$$V = 20y + 5y^2$$

b) Water is running into the trough at a rate of 10 cubic meters per minute. How fast is the water level rising when the water is 0.5 meter deep?

$$\frac{dV}{dt} = 20\frac{dy}{dt} + 10y\frac{dy}{dt}$$

when $y = 3\text{m}$

$$10 = 20\frac{dy}{dt} + 10(0.5)\frac{dy}{dt}$$

$$\frac{dy}{dt} = 0.4\text{m}/\text{min}$$

SOLUTIONS

3. A trough filled with water is 2 m long and has a cross section in the shape of an isosceles trapezoid 30 cm wide at the bottom, 60 cm wide at the top, and a height of 50 cm., if the trough leaks water at the rate of 2000 cm³/min, how fast is the water level falling when the water is 20 cm deep?

$V = \frac{1}{2}(a+b)h \cdot l$

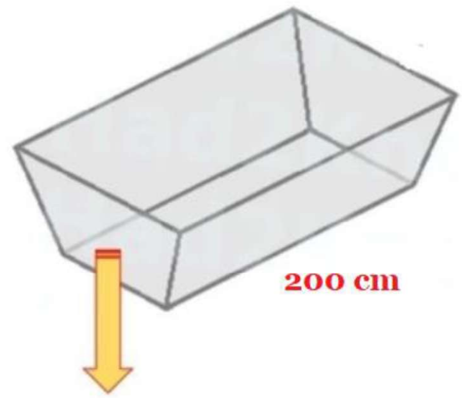
$V = \frac{1}{2}[30+(30+2x)]y(200)$

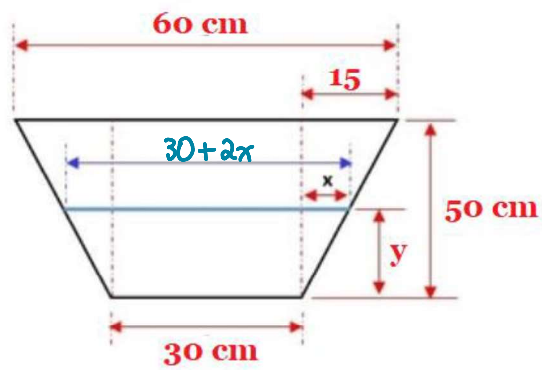
$V = 6000y + 200xy$

From similar triangles: $\frac{x}{y} = \frac{15}{50}$

$x = 0.3y$

$V = 6000y + 60y^2$





$\frac{dV}{dt} = 6000\frac{dy}{dt} + 120y\frac{dy}{dt}$

when $y = 20\text{cm}$

$-2000 = 6000\frac{dy}{dt} + 120(20)\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{-2000}{(6000+2400)}$

$= -0.238$ cm/min

4. A trough is 12 feet long and 3 feet across the top. Its ends are isosceles triangles with altitudes of 3 feet

a) If water is being pumped into the trough at a rate of 2 cubic feet per minute, how fast is the water level rising when the depth is 1 foot?

$V = \frac{1}{2}bhl \quad (b = h)$

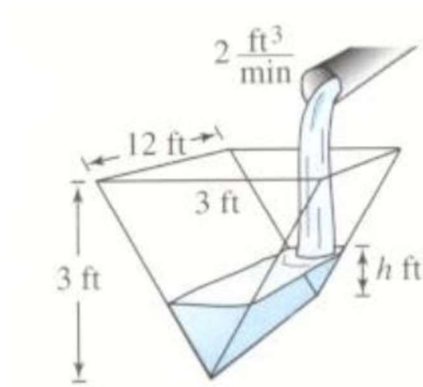
$= \frac{1}{2}h^2(12)$

$V = 6h^2$

$\frac{dv}{dt} = 12h\left(\frac{dh}{dt}\right)$

$2 = 12(1)\left(\frac{dh}{dt}\right)$

$\frac{dh}{dt} = \frac{1}{6}$ ft / min



b) If the water is rising at a rate of 3/8 inch per minute when h = 2, determine the rate at which the water is being pumped into the trough

$$\frac{dh}{dt} = \frac{3}{8}\ \text{in}/\text{min}$$

$$\frac{dh}{dt} = \frac{3}{8}\left(\frac{1}{12}\right)\text{ft}/\text{min}$$

$$V = 6h^2$$

$$\frac{dv}{dt} = 12h \times \frac{dh}{dt}$$

$$= 12(2)\left(\frac{1}{32}\right)$$

$$\frac{dv}{dt} = \frac{3}{4}\ \text{ft}^3/\text{min}$$

RELATED RATES QUESTIONS

Part I

1. Two cars start moving from the same point. One travels south at 60 km/h and the other travels west at 25 km/h. At what rate is the distance between the cars increase two hours later?[65 km/h]

2. Car A is traveling west at 50 km/h and car B is traveling north at 60 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 km from the intersection and car B is 0.4 km from the intersection?[-78km/h]

3. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides from the wall at a rate of 0.2m/s. How long is the ladder?[5 m]

4. How fast is the area of a circle increasing when the circle's radius is 2 m and growing at a rate of 5 m/min? [20π m²/min]

5. Gas is escaping from a spherical weather balloon at a rate of 50cm³/minute. At what rate is the surface area (S) shrinking when the radius is 15 centimeters? [$-\frac{20}{3}$ cm²/min]

6. A square is expanding so that its area increases at 10 cm²/min. How fast is the side length increasing when the area is 52 cm²? [$\frac{5\sqrt{13}}{26}$ cm/min]

7. Josh is 200 m directly north of a buoy, and swims towards it at a rate of 1.2 m/s. Kim is 240 m directly east of the same buoy and swims towards it at a rate of 1.4 m/s. Determine the rate at which the distance between the two swimmers is changing 100 seconds after they start. [-1.84 m/s]

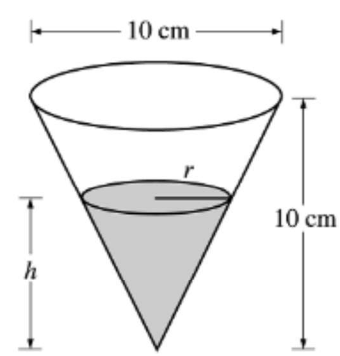
Part II

1. Water is pouring into a conical tank at the rate of 8 cubic meter per minute. If the height of the tank is 12 m and the radius of its circular opening is 6 m, how fast is the water level rising when the water is 4 m deep?

2. Sand is falling off a conveyor belt at a rate of 12 cubic meter per minute into a conical pile. The diameter of the pile is four times the height. At what rate is the height of the pile changing when the pile is 10 m high?

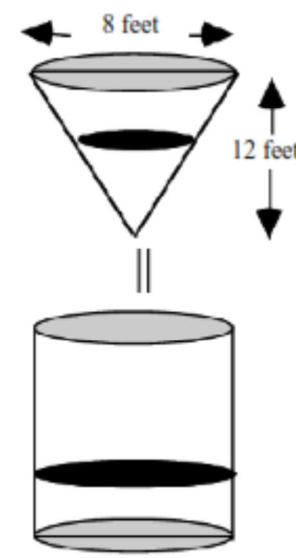
3. A water tank has the shape of an inverted circular cone with a base diameter of 8 m and a height of 12 m.
   a) If the tank is being filled with water at the rate of 5 m³/min, at what rate is the water level increasing when the water is 5 m deep?
   b) If the water tank is full of water and being drained at the rate of 7 m³/min, at what rate is the water level decreasing when the water is 7 m deep?

4. A container has the shape of an open right circular cone, as shown in the figure below. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/h.



   a) Find the volume V of water in the container when h = 5 cm. Indicate units of measure.
   b) Find the rate of change of the volume of water in the container, with respect to time, when h= 5 cm. Indicate units of measure.
   c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

5. As shown in the figure below, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h - 12)$ feet per minute.
   a) Write an expression for the volume of water in the conical tank as a function of h.
   b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.
   c) Let y be the depth, in feet, of water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.



Part II -Answers

1. $\dfrac{2}{\pi}$ m/min 2. $\dfrac{3}{100\pi}$ m/min

3. a) increasing at a rate of 0.573 m/min
   b) decreasing at a rate of 0.409 m/min

4. a) $\dfrac{125\pi}{12}\text{cm}^3$ b) $-\dfrac{15\pi}{8}\text{cm}^3/\text{hr}$ c) $k = \dfrac{dh}{dt} = -\dfrac{3}{10}$

5. a) $V = \dfrac{\pi}{27}h^3$ b) $\dfrac{dV}{dt} = -9\pi\ \text{ft}^3/\text{min}$ c) $\dfrac{dy}{dt} = \dfrac{9}{400}$ ft/min

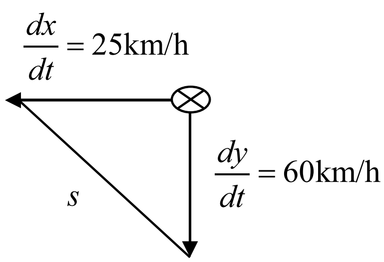
SOLUTIONS

MCV4UZ Related Rates Worksheet Date: ___________

SOLUTIONS

Part I

1. Two cars start moving from the same point. One travels south at 60 km/h and the other travels west at 25 km/h. At what rate is the distance between the cars increase two hours later?[65 km/h]



After 2hours: $x = 2\times 25 = 50km$

$$y = 2\times 60 = 120km$$

$$s = \sqrt{x^2 + y^2}$$

$$s = \sqrt{50^2 + 120^2}$$

$$s = 130\text{km}$$

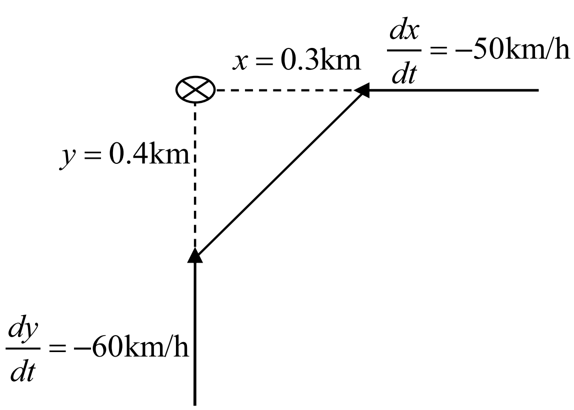
$$x^2 + y^2 = s^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}s\frac{ds}{dt}$$

$$(50)(25) + (120)(60) = (130)\frac{ds}{dt}$$

$$\frac{ds}{dt} = 65\text{km/h}$$

2. Car A is traveling west at 50 km/h and car B is traveling north at 60 km/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 km from the intersection and car B is 0.4 km from the intersection?[-78km/h]



$$s = \sqrt{x^2 + y^2}$$

$$s = \sqrt{0.3^2 + 0.4^2}$$

$$s = 0.5\text{km}$$

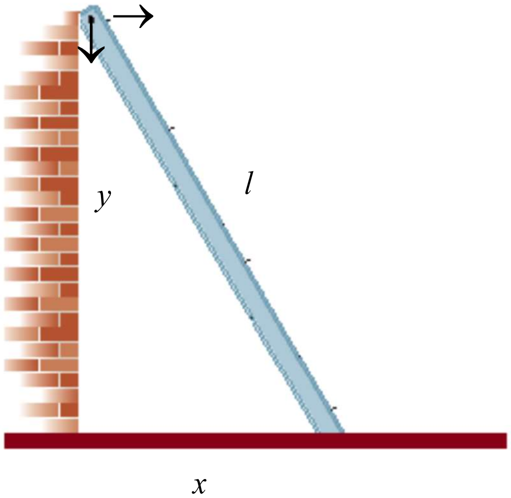
$$x^2 + y^2 = s^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = \cancel{2}s\frac{ds}{dt}$$

$$(0.3)(-50) + (0.4)(-60) = (0.5)\frac{ds}{dt}$$

$$\frac{ds}{dt} = -78\text{km/h}$$

3. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides from the wall at a rate of 0.2m/s. How long is the ladder?[5 m]



$$\frac{dy}{dt} = -0.15\text{m/s}$$

$$\text{when } x=3\text{ m}, \frac{dx}{dt} = 0.2\text{m/s}$$

$$x^2 + y^2 = l^2$$

$$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$$

$$3(0.2) + y(-0.15) = 0$$

$$y = 4m$$

$$l = \sqrt{x^2 + y^2}$$

$$l = \sqrt{3^2 + 9^2}$$

$$l = \sqrt{25}$$

$$l = 5m$$

4. How fast is the area of a circle increasing when the circle's radius is 2 m and growing at a rate of 5 m/min? [20π m²/min]

$$A = \pi r^2$$

$$\frac{dr}{dt} = 5\text{m/min}$$

$$\frac{dA}{dt} = ?\ when\ r = 2\ m$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(2)(5)$$

$$= 20\pi\,\text{m}^3/\text{min}$$

5. Gas is escaping from a spherical weather balloon at a rate of 50cm³/minute. At what rate is the surface area (S) shrinking when the radius is 15 centimeters? [$-\frac{20}{3}$ cm²/min]

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = -50\,\text{cm}^3/\text{min}$$

$$\frac{dS}{dt} = ?\ when\ r = 15\ m$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

$$-50 = 4\pi(15)^2\frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-1}{18\pi}\text{cm/min}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi(15)\left(\frac{-1}{18\pi}\right)$$

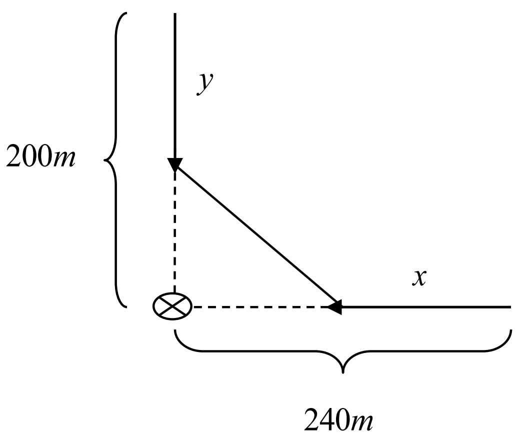
$$\frac{dS}{dt} = \frac{-20}{3}\text{cm}^2/\text{min}$$

6. A square is expanding so that its area increases at 10 cm^2/min. How fast is the side length increasing when the area is 52 cm^2? [$\frac{5\sqrt{13}}{26}$ cm/min]

$$A = x^2$$
$$52 = x \rightarrow x = 2\sqrt{13}\text{ cm}$$
$$\frac{dA}{dt} = 10\,\text{cm}^2/\text{min}$$

$$\frac{dA}{dt} = 2x\frac{dx}{dt}$$
$$10 = 2\left(2\sqrt{13}\right)\frac{dx}{dt}$$
$$\frac{dx}{dt} = \frac{10}{4\sqrt{13}}\text{ cm/min}$$
$$= \frac{5\sqrt{13}}{26}\text{ cm/min}$$

7. Josh is 200 m directly north of a buoy, and swims towards it at a rate of 1.2 m/s. Kim is 240 m directly east of the same buoy and swims towards it at a rate of 1.4 m/s. Determine the rate at which the distance between the two swimmers is changing 100 seconds after they start. [-1.84 m/s]



After 100 seconds: $x = 100 \times 1.4 = 140m$

$$y = 100 \times 1.2 = 120m$$
$$s = \sqrt{(240-x)^2 + (200-y)^2}$$
$$s = \sqrt{(240-140)^2 + (200-120)^2}$$
$$s = \sqrt{(100)^2 + (80)^2}$$
$$s = 20\sqrt{41}\text{ m}$$

$$(240-x)^2 + (200-y)^2 = s^2$$
$$\cancel{2}(240-x)\left(-\frac{dx}{dt}\right) + \cancel{2}(200-y)\left(-\frac{dy}{dt}\right) = \cancel{2}s\frac{ds}{dt}$$
$$(100)(-1.4) + (80)(-1.2) = \left(20\sqrt{41}\right)\frac{ds}{dt}$$
$$\frac{ds}{dt} = -1.84\text{m/s}$$

Part II

1. Water is pouring into a conical tank at the rate of 8 cubic meter per minute. If the height of the tank is 12 m and the radius of its circular opening is 6 m, how fast is the water level rising when the water is 4 m deep?

$$V=\frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt}=8\text{m}^3/\text{min}$$

$$\frac{dh}{dt}=?\rightarrow when\ h=4m$$

$$\frac{h}{12}=\frac{r}{6}\rightarrow r=\frac{1}{2}h$$

$$V=\frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

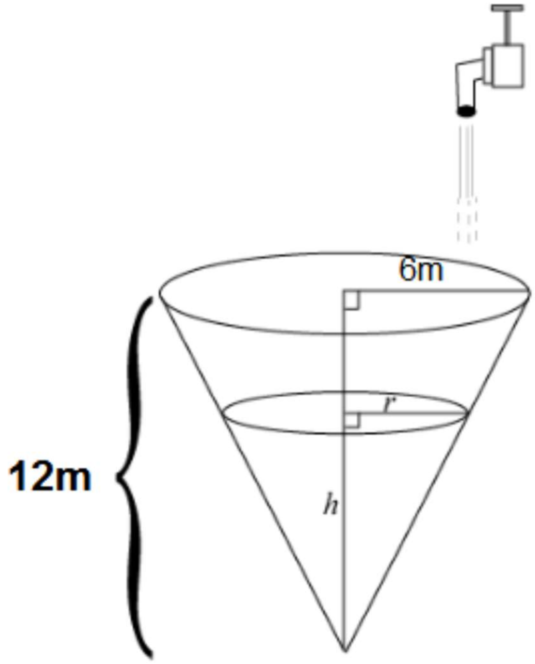
$$V=\frac{\pi}{12}h^3$$

$$\frac{dV}{dt}=\frac{\pi}{4}h^2\frac{dh}{dt}$$

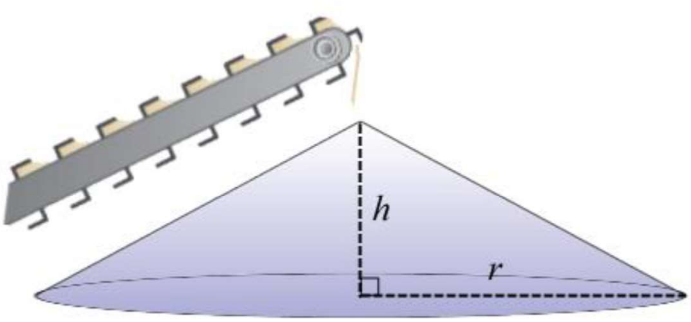
$$8=\frac{\pi}{4}(4)^2\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{2}{\pi}\text{m/min}$$

The water level rising at rate 2/π m/min when the water is 4 m deep



2. Sand is falling off a conveyor belt at a rate of 12 cubic meter per minute into a conical pile. The diameter of the pile is four times the height. At what rate is the height of the pile changing when the pile is 10 m high?



$$V=\frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt}=12\text{m}^3/\text{min}$$

$$\frac{dh}{dt}=?\rightarrow when\ h=10m$$

$$2r=4h\rightarrow r=2h$$

$$V=\frac{1}{3}\pi(2h)^2 h$$

$$V=\frac{4\pi}{3}h^3$$

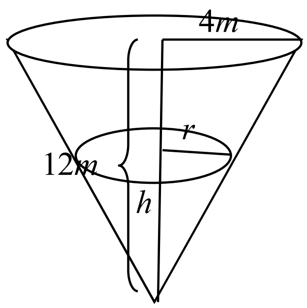
$$\frac{dV}{dt}=4\pi h^2\frac{dh}{dt}$$

$$12=4\pi(10)^2\frac{dh}{dt}$$

$$\frac{dh}{dt}=\frac{3}{100\pi}\text{m/min}$$

3. A water tank has the shape of an inverted circular cone with a base diameter of 8 m and a height of 12 m.
   a. If the tank is being filled with water at the rate of 5 m^3/min, at what rate is the water level increasing when the water is 5 m deep? [increasing at a rate of 0.573 m/min]
   b. If the water tank is full of water and being drained at the rate of 7 m^3/min, at what rate is the water level decreasing when the water is 7 m deep? [decreasing at a rate of 0.409 m/min]

a)



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = 5\text{m}^3/\text{min}$$

$$\frac{dh}{dt} = ? \rightarrow when\ h = 5m$$

$$\frac{r}{4} = \frac{h}{12} \rightarrow r = \frac{1}{3}h$$

$$V = \frac{1}{3}\pi r^2 h \xrightarrow{r=\frac{2}{3}h} V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$V = \frac{1}{27}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

$$5 = \frac{1}{9}\pi(5)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{5\pi} \doteq 0.573 \text{ m/min}$$

b)

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = -7\text{m}^3/\text{min}$$

$$\frac{dh}{dt} = ? \rightarrow when\ h = 7m$$

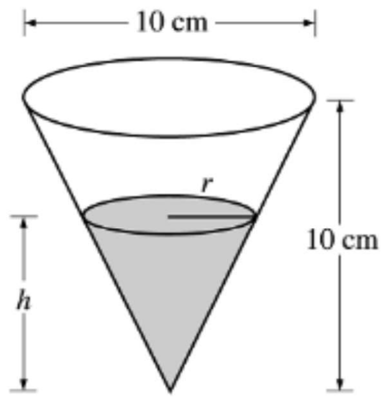
$$V = \frac{1}{27}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$$

$$-7 = \frac{1}{9}\pi(7)^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{9}{7\pi} \doteq -0.409 \text{ m/min}$$

4. A container has the shape of an open right circular cone, as shown in the figure below. The height of the container is 10 cm and the diameter of the opening is 10 cm. Water in the container is evaporating so that its depth h is changing at the constant rate of $\frac{-3}{10}$ cm/h.



(a) Find the volume V of water in the container when $h = 5$ cm. Indicate units of measure.

(b) Find the rate of change of the volume of water in the container, with respect to time, when $h = 5$ cm. Indicate units of measure.

(c) Show that the rate of change of the volume of water in the container due to evaporation is directly proportional to the exposed surface area of the water. What is the constant of proportionality?

a)

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dh}{dt} = \frac{-3}{10}\text{cm/hr}$$

$$\frac{dV}{dt} = ? \rightarrow when\ h = 5\,cm$$

$$\frac{r}{5} = \frac{h}{10} \rightarrow r = \frac{1}{2}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h$$

$$V = \frac{\pi}{12}h^3 \xrightarrow{h=5} V = \frac{125\pi}{12}\text{cm}^3\text{/hr}$$

b)

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{4}(5)^2\left(\frac{-3}{10}\right)$$

$$\frac{dV}{dt} = \frac{-15\pi}{8}\text{cm}^3\text{/hr}$$

c)

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{r}{5} = \frac{h}{10} \rightarrow h = 2r \Leftrightarrow \frac{dh}{dt} = 2\frac{dr}{dt}$$

$$V = \frac{1}{3}\pi r^2(2r)$$

$$V = \frac{2}{3}\pi r^3$$

$$\frac{dV}{dt} = 2\pi r^2\frac{dr}{dt}$$

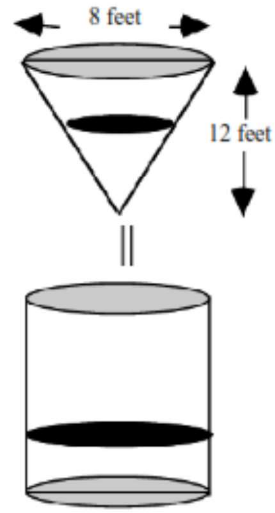
$$\frac{dV}{dt} = \cancel{2}\pi r^2\left(\frac{1}{\cancel{2}}\times\frac{dh}{dt}\right)$$

$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt} \Leftrightarrow \underbrace{\frac{\left(\frac{dV}{dt}\right)}{\left(\pi r^2\right)}}_{SA} = \frac{dh}{dt}$$

$$\frac{\left(\frac{dV}{dt}\right)}{SA} = -\frac{3}{10}\text{cm/hr}$$

SOLUTIONS

5. As shown in the figure, water is draining from a conical tank with height 12 feet and diameter 8 feet into a cylindrical tank that has a base with area 400π square feet. The depth h, in feet, of the water in the conical tank is changing at the rate of $(h - 12)$ feet per minute.



(a) Write an expression for the volume of water in the conical tank as a function of h.

(b) At what rate is the volume of water in the conical tank changing when $h = 3$? Indicate units of measure.

(c) Let y be the depth, in feet, of water in the cylindrical tank. At what rate is y changing when $h = 3$? Indicate units of measure.

a)

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dh}{dt} = (h - 12) \text{ ft/min}$$

$$\frac{dV}{dt} = ? \rightarrow \text{when } h = 3\ ft$$

$$\frac{r}{4} = \frac{h}{12} \rightarrow r = \frac{1}{3}h$$

$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$V = \frac{\pi}{27}h^3 \quad \text{ft}^3/\text{min}$$

b)

$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{9}(3)^2(-9)$$

$$\frac{dV}{dt} = -9\pi \quad \text{ft}^3/\text{min}$$

c)

$$V = 400\pi y$$

$$\frac{dV}{dt} = 400\pi\frac{dy}{dt}$$

$$9\pi = 400\pi\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{9}{400} \text{ft/min}$$

Relax.....Review

1. Differentiate the following:

a) $m(t)=\frac{\pi}{3}t^{3}-3t^{-5}+4\pi^{2}$

b) $f(x)=\left(1+x^{\frac{3}{4}}\right)\left(\sqrt{x+\sqrt{x}}\right)$ (do not simplify)

c) $g(x)=\frac{x^{3}+4}{x^{3}-3x+1}$

d) $y=\frac{(3x^{2}-1)^{-4}}{(x^{3}-2x)^{-5}}$

2. If $g(x)=\frac{1}{2x-4}+\sqrt{x}$, find $g'''(4)$.

3. For what value(s) of *k* will the line $2x-3y+k=0$ be normal to $y=\sqrt{3x^{2}+4}$?

4. Find the rate of change for $s(t)=\left(\frac{t-\pi}{t-10\pi}\right)^{\frac{1}{3}}$ at $t=2\pi$. Leave final answer in terms of π.

5. Two tangents are drawn from the point (2,6) to the graph of $y=-x^{2}-5x+4$. Determine the coordinates of the point(s) where the tangents touch the graph.

6. For what values of *a* and *b* will the parabola $y=x^{2}+ax+b$ be tangent to the curve $y=x^{3}+5x$ at the point x=1?

7. A 1500-L tank leaks water so that the volume of water, in litres, remaining after *t* days, $0\le t\le 15$, is represented by $V(t)=1500\left(1-\frac{t}{15}\right)^{2}$. How rapidly is the water leaking when the tank is $\frac{1}{9}$ full? Round final answers to 2 decimal places.

8. Find the values of *x* so that the tangent to $f(x)=\frac{3}{\sqrt[3]{x}}$ is parallel to the line $x+16y+3=0$.

9. Find **a** and **b** so that the line y = - ax +4 is tangent to the graph of y= ax³ + bx at $x=1$.

10. Find the constant value(s) of ***k*** such that the equation of tangent to the curve $f(x)=\sqrt{1-kx^{2}}$ at $x=1$ is parallel to the line $3x-2y+1=0$.

11. Two lines drawn from point $A\left(0,\frac{7}{4}\right)$ are tangent to the parabola $y=1-x^{2}$ at P and Q. Find the area of triangle APQ.

12. Let f be a function given by $f(x)=\frac{ax^{2}+b}{x+c}$ and that has the following properties

$\lim_{x\to -1^{-}} f(x)=\infty,\quad f'(0)=2,\quad f''(0)=-2.$ Determine the values of a, b and c

13. Let $f(x)=\sqrt{ax^{2}+b}$. Find values of a and b such that the linear equation $7x+2y=5$ is tangent to $f(x)$ at $x=-1$.

14. Find the area of the triangle determined by the coordinate axes and the tangent to the curve $xy=1$ at $x=1$.

15. Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

16. The equation of the tangent to $y = 2x^2 - 1$ at the point where $x = 1$, is $4ax - y = 2b^2 + 1$. Find the values of a and b.

17. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ so that the line $y = -4x + 1$ is tangent to the graph of $y = \frac{a}{x} + \frac{b}{x+1}$ at $x = 1$.

18. The curve $y = 2x^3 + ax + b$ has a tangent with slope 10 at the point (-2, 33). Find the values of a and b.

19. The position of an object moving along a straight line is described by the function $s(t) = -t^3 + 4t^2 - 10 \quad for\ t \geq 0.$
    (a) Is the object moving away from or towards its initial position when t = 3?
    (b) Is the object speeding up or slowing down when t = 3?

20. A position function of an object is given by: $s(t) = t^3 - 6t^2 + 8t \quad ,t \geq 0$
    (a) Determine the velocity function for the object.
    (b) Identify the point(s) where the object is at rest.
    (c) Identify the point(s) where the acceleration is zero.
    (d) Determine the equation of the acceleration function.
    (e) For which intervals is the acceleration negative? Positive?
    (f) Determine the intervals for which the object is speeding up and slowing down.

21. A north-south highway intersects an east-west highway at a point P. An automobile crosses P at 10:00 AM, travelling east at a constant speed of 20 km / hr. At the same instant, another automobile is 2km north of P , travelling south at 50km/hr. Find the time at which they are closest to each other, and approximate the minimum distance between the automobile.

22. At noon a car is driving west at 55 km/h. At the same time, 15 km due north another car is driving south at 85 km/h. At what rate the distance between two cars changing 4 hours later?

23. How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s ?

24. A water tank is in the shape of an inverted right circular cone with top radius 10 m and depth 8 m. Water is flowing in at a rate of 0.1 m^3 /min. How fast is the depth of water in the tank increasing 3 minutes later?

25. A ladder 25 m long is leaning against a house. The base of the ladder is pulled away from the wall at a rate of 2 meter per second. How fast is the top of the ladder moving down the wall when the base of the ladder is 12 meter from the wall? Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 m from the wall.

1. Differentiate the following:

a) $m(t)=\frac{\pi}{3}t^3-3t^{-5}+4\pi^2$

$$m'(t)=\pi t^2+\frac{15}{t^6}$$

b) $f(x)=\left(1+x^{\frac{3}{4}}\right)\left(\sqrt{x+\sqrt{x}}\right)$ (do not simplify)

$$f'(x)=\left(\frac{3}{4}x^{\frac{-1}{4}}\right)\left(\sqrt{x+\sqrt{x}}\right)+\left(\frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}\right)\left(1+x^{\frac{3}{4}}\right)$$

c) $g(x)=\frac{x^3+4}{x^3-3x+1}$

$$g'(x)=\frac{3x^2\left(x^3-3x+1\right)-\left(3x^2-3\right)\left(x^3+4\right)}{\left(x^3-3x+1\right)^2}$$

$$=\frac{\cancel{3x^5}-9x^3+3x^2\cancel{-3x^5}-12x^2+3x^3+12}{\left(x^3-3x+1\right)^2}$$

$$=\frac{-3\left(2x^3+3x^2-4\right)}{\left(x^3-3x+1\right)^2}$$

d) $y=\frac{\left(3x^2-1\right)^{-4}}{\left(x^3-2x\right)^{-5}}$

$$y=\left(3x^2-1\right)^{-4}\left(x^3-2x\right)^5$$

$$y'=-4\left(3x^2-1\right)^{-5}(6x)\left(x^3-2x\right)^5+5\left(x^3-2x\right)^4\left(3x^2-2\right)\left(3x^2-1\right)^{-4}$$

$$y'=\left(3x^2-1\right)^{-5}\left(x^3-2x\right)^4\left[-24x\left(x^3-2x\right)+5\left(3x^2-1\right)\left(3x^2-2\right)\right]$$

$$y'=\left(3x^2-1\right)^{-5}\left(x^3-2x\right)^4\left[-24x^4+48x^2+45x^4-45x^2+10\right]$$

$$y'=\frac{\left(x^3-2x\right)^4\left(21x^4+3x^2+10\right)}{\left(3x^2-1\right)^5}$$

SOLUTIONS

2. If $g(x)=\frac{1}{2x-4}+\sqrt{x}$, find $g'''(4)$.

$g(x)=\frac{1}{2}(x-2)^{-1}+x^{\frac{1}{2}}$

$g'(x)=-\frac{1}{2}(x-2)^{-2}+\frac{1}{2}x^{-\frac{1}{2}}$

$g''(x)=(x-2)^{-3}-\frac{1}{4}x^{-\frac{3}{2}}$

$g'''(x)=-3(x-2)^{-4}+\frac{3}{8}x^{-\frac{5}{2}}$

$g'''(4)=-3(2)^{-4}+\frac{3}{8}(4)^{-\frac{5}{2}}$

$g'''(4)=\frac{-45}{256}$

3. For what value(s) of k will the line $2x-3y+k=0$ be normal to $y=\sqrt{3x^2+4}$?

$y'=\frac{3x}{\sqrt{3x^2+4}}$

$m=\frac{2}{3}\rightarrow m_{\perp}=-\frac{3}{2}$

$\Rightarrow \frac{\cancel{3}x}{\sqrt{3x^2+4}}=-\frac{\cancel{3}}{2}$

$\sqrt{3x^2+4}=-2x \quad (x<0)$

$3x^2+4=4x^2$

$x^2=4$

$x=\pm 2 \xrightarrow{x<0} x=-2$

$y=\sqrt{3(-2)^2+4}$

$y=4$

sub. (-2,4) into $2x-3y+k=0$ to get k.

$-4-12+k=0$

$k=16$

4. Find the rate of change for $s(t)=\left(\frac{t-\pi}{t-10\pi}\right)^{\frac{1}{3}}$ at $t=2\pi$. Leave final answer in terms of π.

$s'(t)=\frac{1}{2}\left(\frac{t-\pi}{t-10\pi}\right)^{-\frac{2}{3}}\left(\frac{\cancel{t}-10\pi-\cancel{t}+\pi}{(t-10\pi)^2}\right)$

$s'(2\pi)=\frac{1}{2}\left(\frac{2\pi-\pi}{2\pi-10\pi}\right)^{-\frac{2}{3}}\left(\frac{-9\pi}{(2\pi-10\pi)^2}\right)$

$s'(2\pi)=\frac{1}{2}\left(\frac{\cancel{\pi}}{-8\cancel{\pi}}\right)^{-\frac{2}{3}}\left(\frac{-9\pi}{64\pi^2}\right)$

$=\frac{-3}{16\pi}$

5. Two tangents are drawn from the point (2,6) to the graph of $y=-x^2-5x+4$. Determine the coordinates of the point(s) where the tangents touch the graph.

Let point A be $(x, -x^2-5x+4)$ and B be $(2,6)$

$y' = m_{AB}$

$-2x-5 = \dfrac{6+x^2+5x-4}{2-x} \leftarrow \dfrac{\Delta y}{\Delta x}$

$(-2x-5)(2-x) = x^2+5x+2$

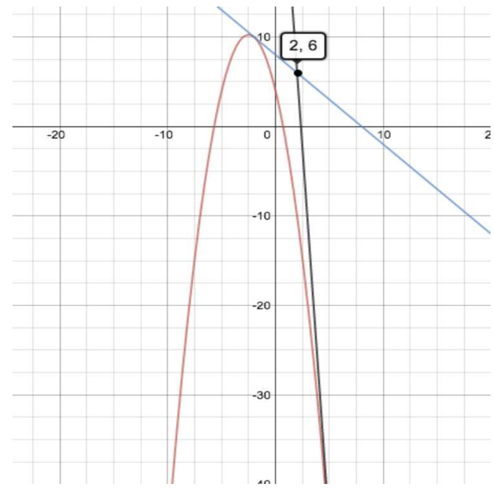
$-4x+2x^2-10+5x = x^2+5x+2$

$x^2-4x-12=0$

$(x-6)(x+2)=0$

$x=6$, $x=-2$

$y=-62$, $y=10$



6. For what values of a and b will the parabola $y=x^2+ax+b$ be tangent to the curve $y=x^3+5x$ at the point x=1?

Sub $x=1$ into both functions

$y=(1)^2+a(1)+b$ $\quad$ $y_2=(1)^3+5(1)$

$y_1=1+a+b$ $\quad$ $=6$

$\therefore 1+a+b=6$

$a+b=5$ ①

Sub $x=1$ into both derivatives:

$y_1'=2x+a$ $\quad$ $y_2'=3x^2+5$

$y_1'=2(1)+a$ $\quad$ $y_2'=3(1)^2+5$

$=2+a$ $\quad$ $y_2'=8$

$\therefore 2+a=8$

$a=6$

Sub $a=6$ into eq'n ①

$(6)+b=5$

$b=-1$

7. A 1500-L tank leaks water so that the volume of water, in litres, remaining after t days, $0 \le t \le 15$, is represented by $V(t)=1500\left(1-\frac{t}{15}\right)^2$. How rapidly is the water leaking when the tank is $\frac{1}{9}$ full? Round final answers to 2 decimal places.

$\frac{1}{9}\times 1500 = 1500\left(1-\frac{t}{15}\right)^2$

$\frac{1}{9} = \left(1-\frac{t}{15}\right)^2$

$\pm\frac{1}{3} = 1-\frac{t}{15}$

$\frac{t}{15} = 1-\frac{1}{3}$ OR $\frac{t}{15} = 1+\frac{1}{3}$

$\frac{t}{15} = \frac{2}{3}$ $\quad$ $\frac{t}{15} = \frac{4}{3}$

$t = 10$ days $\quad$ $t = 20$ days

$V'(t) = 3000\left(1-\frac{t}{15}\right)\left(\frac{-1}{15}\right)$

$V'(10) = 3000\left(1-\frac{10}{15}\right)\left(\frac{-1}{15}\right)$

$= -\frac{200}{3}$

$= -66.67$ L / days

8. Find the values of x so that the tangent to $f\left(x\right)=\dfrac{3}{\sqrt[3]{x}}$ is parallel to the line $x + 16y + 3 = 0$.

$$\mathbf{f\left(x\right)=3x^{-\frac{1}{3}}}$$

$$\mathbf{f'\left(x\right)=-x^{-\frac{4}{3}}}$$

$$\mathbf{m_{\perp}=-\frac{1}{16}}$$

$$\Rightarrow$$

$$\mathbf{-x^{-\frac{4}{3}}=-\frac{1}{16}}$$

$$\mathbf{x^{\frac{4}{3}}=16}$$

$$\mathbf{x^{4}=16^{3}}$$

$$\mathbf{x=\pm 8}$$

9. Find **a** and **b** so that the line $y=-\mathbf{a}x+4$ is tangent to the graph of $y=ax^3+bx$ at $x=1$.

$$\mathbf{f'\left(x\right)=3ax^2+bx}$$

$$\mathbf{m_T=f'\left(1\right)=3a+b}$$

$$\mathbf{y=-ax+4 \quad \uparrow m_T=-a}$$

$$\mathbf{3a+b=-a \quad \uparrow b=-4a \qquad (1)}$$

$$\left.\begin{array}{l}\mathbf{f\left(1\right)=a+b}\\ \mathbf{y=-a+4}\end{array}\right\rangle \mathbf{a+b=-a+4 \xrightarrow{b=-4a} -2a=4}$$

$$\mathbf{a=-2 \quad , \quad b=8}$$

$\frac{dy}{dx}=3ax^2+b$

$\left.\frac{dy}{dx}\right|_{x=1}=3a+b$

$\left.\frac{dy}{dx}\right|_{x=1}=-a$

$\therefore 3a+b=-a$

$b=-4a$ ①

At $x=1$,

$y_1=a(1)^3+b(1)$

$y_1=a+b$

$y_2=-a+4$

$y_1=y_2$, sub ①

$a+(-4a)=-a+4$

$-2a=4$

$a=-2$

sub into ①

$b=-4(-2)$

$=8$

10. Find the constant value of **k** such that the equation of tangent to the curve $f\left(x\right)=\sqrt{1-kx^2}$ at x=1 is parallel to the line $3x-2y+1=0$

$$\mathbf{y=\frac{3}{2}x+\frac{1}{2}}$$

$$\mathbf{f'\left(x\right)=\frac{-\cancel{2}kx}{\cancel{2}\sqrt{1-kx^2}}}$$

$$\mathbf{f'\left(1\right)=\frac{3}{2}}$$

$$\mathbf{\frac{-k}{\sqrt{1-k}}=\frac{3}{2}}$$

$$\mathbf{-2k=3\sqrt{1-k} \quad \left(k<0\right)}$$

$$\mathbf{4k^2=9\left(1-k\right)}$$

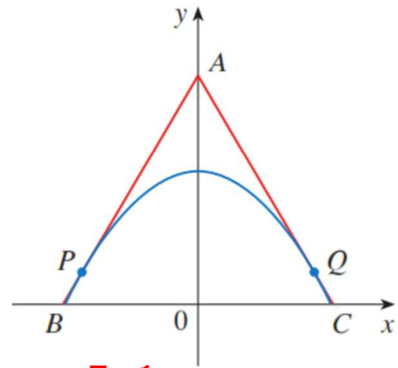
$$\mathbf{4k^2+9k-9=0}$$

$$\mathbf{\left(4k-3\right)\left(k+3\right)=0}$$

$$\mathbf{k=\frac{3}{4} \ , \ \boxed{k=-3}}$$

SOLUTIONS

11. Two lines drawn from point $A\left(0,\frac{7}{4}\right)$ are tangent to the parabola $y=1-x^2$ at P and Q. Find the area of triangle APQ.



$$\text{Let } Q\left(x,1-x^2\right), A\left(0,\frac{7}{4}\right)$$

$$y' = m_{AQ}$$

$$-2x = \frac{1-x^2-\frac{7}{4}}{x}$$

$$-2x^2 = \frac{-3}{4} - x^2$$

$$\frac{3}{4} = x^2$$

$$x = \pm\frac{\sqrt{3}}{2} \quad \uparrow P\left(-\frac{\sqrt{3}}{2},\frac{1}{4}\right), Q\left(\frac{\sqrt{3}}{2},\frac{1}{4}\right)$$

$$|PQ| = \text{base} = \sqrt{3} \quad , \quad \text{height} = \frac{7}{4}-\frac{1}{4}$$

$$= \frac{3}{2}$$

$$\text{Area}_{\measuredangle APQ} = \frac{\text{base}\times\text{height}}{2}$$

$$= \frac{\sqrt{3}\times\frac{3}{2}}{2}$$

$$= \frac{3\sqrt{3}}{4} \text{ units}^2$$

12. Let f be a function given by $f(x)=\dfrac{ax^2+b}{x+c}$ and that has the following properties:

$\lim\limits_{x\to-1^-} f(x)=\infty$, $f'(0)=2$, $f''(0)=-2$. Determine the values of a , b and c .

$$\lim_{x \uparrow 1^-} f(x) = \infty \Rightarrow -1+c=0$$

$$\boxed{c=1}$$

$$f(x)=\frac{ax^2+b}{x+1}$$

$$f'(x)=\frac{2ax(x+1)-\left(ax^2+b\right)}{(x+1)^2}$$

$$f'(0)=2: \ -b=2 \quad \uparrow \boxed{b=-2}$$

$$f'(x)=\frac{ax^2+2ax+2}{(x+1)^2}$$

$$f''(x)=\frac{(2ax+2a)(x+1)^2-2(x+1)\left(ax^2+2ax+2\right)}{(x+1)^4}$$

$$f''(0)=-2: 2a-4=-2$$

$$\boxed{a=1}$$

SOLUTIONS

13. Let $f(x)=\sqrt{ax^2+b}$. Find values of a and b such that the linear equation $7x+2y=5$ is tangent to $f(x)$ at $x=-1$.

$$f'(x)=\frac{ax}{\sqrt{ax^2+b}}$$

$$f'(-1)=-\frac{7}{2}$$

$$\frac{-a}{\sqrt{a+b}}=\frac{-7}{2} \quad (a\geq 0)$$

$$2a=7\sqrt{a+b} \quad (1)$$

$$x=-1: \quad 7(-1)+2y=5$$

$$y=6$$

$$f(-1)=6$$

$$\sqrt{a+b}=6 \quad (2)$$

Sub. (2) into (1):

$$2a=7(6)$$

$$\boxed{a=21} \xrightarrow{\sqrt{a+b}=6} \boxed{b=15}$$

14. Find the area of the triangle determined by the coordinate axes and the tangent to the curve $xy = 1$ at $x = 1$.

$$y=\frac{1}{x} \quad \text{point}(1,1)$$

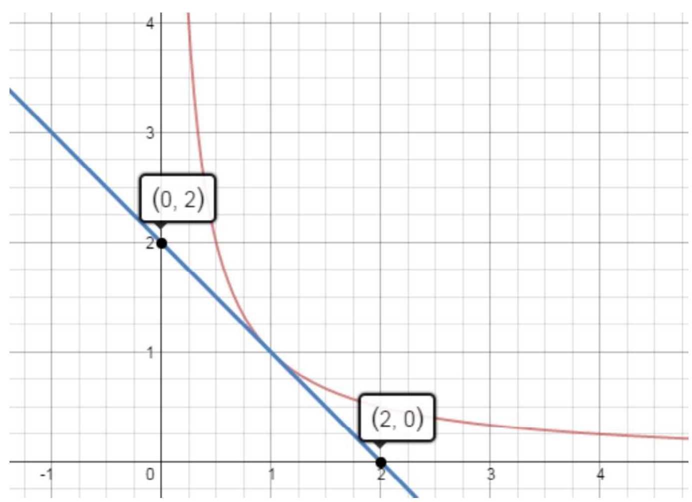
$$y'=-\frac{1}{x^2} \to m_t=-1$$

Equation of tangent line: $y-1=-1(x-1)$

$$y=-x+2$$

x-int(2,0) & y-int: (0,2)

Area of tringle: $\frac{1}{2}(2\times 2)=2 \text{ units}^2$



15. Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where *a* and *b* are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of *a* and *b*.

$$y = ax^{\frac{1}{2}} + bx^{\frac{-1}{2}}$$

$$y = x^{\frac{-1}{2}}(ax + b)$$

$$y' = \frac{-1}{2}x^{\frac{-3}{2}}(ax + b) + ax^{\frac{-1}{2}}$$

$$m_T = \frac{-1}{2}(4)^{\frac{-3}{2}}(4a + b) + a(4)^{\frac{-1}{2}}$$

$$m_T = \frac{-1}{16}(4a + b) + \frac{1}{2}a$$

$$m_T = \frac{1}{4}a - \frac{1}{16}b$$

$$4x + y = 22 \quad \uparrow y = -4x + 22$$

$$m_{\perp} = -4 \quad \uparrow m_T = \frac{1}{4}$$

$$\frac{1}{4}a - \frac{1}{16}b = \frac{1}{4} \quad \uparrow 4a - b = 4 \quad (1)$$

$$y = a\sqrt{x} + \frac{b}{\sqrt{x}} \xrightarrow{\text{at } x=4} y = 2a + \frac{b}{2}$$

$$4x + y = 22 \xrightarrow{\text{at } x=4} y = 6$$

$$2a + \frac{b}{2} = 6$$

$$4a + b = 12 \quad (2)$$

$$\left.\begin{aligned} 4a - b &= 4 \quad (1) \\ 4a + b &= 12 \quad (2) \end{aligned}\right\rangle \; a = 2\,,\; b = 4$$

16. The equation of the tangent to $y = 2x^2 - 1$ at the point where $x = 1$, is $4ax - y = 2b^2 + 1$. Find the values of *a* and *b*.

$$y = 4ax - (2b^2 + 1)$$

$$y = 2x^2 - 1 \rightarrow y' = 4x \rightarrow m_t = 4$$

$$4 = 4a$$

$$\boxed{a = 1}$$

$$\text{At } x = 1, y = 1$$

$$1 = 4 - (2b^2 + 1)$$

$$2b^2 = 2$$

$$\boxed{b = \pm 1}$$

17. Find $\boldsymbol{a}$ and $\boldsymbol{b}$ so that the line $y=-4x+1$ is tangent to the graph of $y = \frac{a}{x}+\frac{b}{x+1}$ at $x=1$.

$$y' = -\frac{a}{x^2} - \frac{b}{(x+1)^2}$$

$$\left.\begin{array}{l} m_T = -a - \frac{b}{4} \\ y = -4x+1 \end{array}\right\rangle -a - \frac{b}{4} = -4 \quad \Rightarrow \quad 4a+b=16 \quad (1)$$

$$\left.\begin{array}{l} \text{at } x=1 \ \ y = a + \frac{b}{2} \\ \text{at } x=1 \ \ y=-4(1)+1 \end{array}\right\rangle a + \frac{b}{2} = -3 \rightarrow 2a+b=-6 \quad (2)$$

$$\left.\begin{array}{l} 4a+b=16 \\ 2a+b=-6 \end{array}\right\rangle \boxed{a=11},\ \boxed{b=-28}$$

18. The curve $y = 2x^3 + ax + b$ has a tangent with slope 10 at the point (-2, 33). Find the values of a and b.

$$y = 2x^3 + ax + b$$
$$y' = 6x^2 + a$$
$$10 = 6(-2)^2 + a$$
$$10 - 24 = a$$
$$\boxed{a=-14}$$

$$33 = 2(-2)^3 - 14(-2) + b$$
$$\boxed{b=21}$$

19. The position of an object moving along a straight line is described by the function

$$s(t) = -t^3 + 4t^2 - 10 \quad \mathit{for} \ t \geq 0.$$

(a) Is the object moving away from or towards its initial position when t = 3?
(b) Is the object speeding up or slowing down when t = 3?

$$s(t) = -t^3 + 4t^2 - 10$$
$$v(t) = -3t^2 + 8t$$
$$a(t) = -6t + 8$$
$$s(3)v(3) = (-1)(-3) > 0$$

$\therefore$ **The object is moving away from the origin**

$$a(3)v(3) = (-10)(-3) > 0$$

$\therefore$ **The object is speeding up when $t = 3$**

20. A position function of an object is given by: $s(t) = t^3 - 6t^2 + 8t \quad , t \geq 0$

(a) Determine the velocity function for the object.

$v(t) = 3t^2 - 12t + 8$

(b) Identify the point(s) where the object is at rest.

$0 = 3t^2 - 12t + 8$

$t = \dfrac{6 \pm \sqrt{12}}{3}$

$t = \dfrac{6 \pm 2\sqrt{3}}{3}$

$t \doteq 3.15$ or $t \doteq 0.86$

(c) Identify the point(s) where the acceleration is zero.

$a(t) = 6t - 12$

$6t - 12 = 0$

$t = 2$

(d) Determine the equation of the acceleration function.

$a(t) = 6t - 12$

(e) For which intervals is the acceleration negative? Positive?

```
  -       +
|-----|----->
0     2      ∞
```

The object is deceleration when $0 \leq t < 2$ and it's accelerating when $t > 2$

(f) Determine the intervals for which the object is speeding up and slowing down.

$a(t)v(t) = 6(t-2)(3t^2 - 12t + 8)$

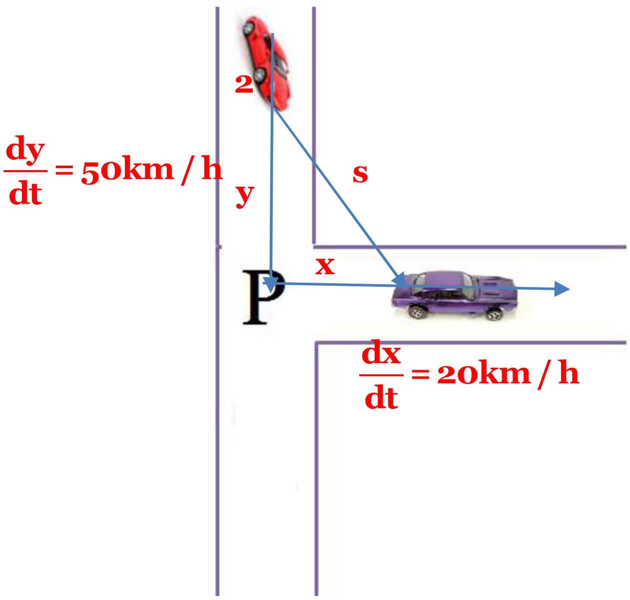
```
     -          +            -            +
|--------|---------|---------|-------->
```

$0 \qquad \dfrac{6-2\sqrt{3}}{3} \qquad 2 \qquad \dfrac{6-2\sqrt{3}}{3} \qquad \infty$

The object is speeding up when $\dfrac{6-2\sqrt{3}}{3} < t < 2$ or $t > \dfrac{6+2\sqrt{3}}{3}$

The object is slowing down when $0 \leq t < \dfrac{6-2\sqrt{3}}{3}$ or $2 < t < \dfrac{6+2\sqrt{3}}{3}$

21. A north-south highway intersects an east-west highway at a point P. An automobile crosses P at 10:00 AM, travelling east at a constant speed of 20 km / hr. At the same instant, another automobile is 2km north of P , travelling south at 50km/hr. Find the time at which they are closest to each other, and approximate the minimum distance between the automobile.



$$(2-y)^2+x^2=s^2$$

$$-2(2-y)\frac{dy}{dt}+2x\frac{dx}{dt}=2s\frac{ds}{dt}$$

Let $s=0$, since $y=50t$ and $x=20t$ we get

$$(2-y)\frac{dy}{dt}=x\frac{dx}{dt}$$

$$(2-50t)(50)=(20t)(20)$$

$$10-250t=40t$$

$$10=290t$$

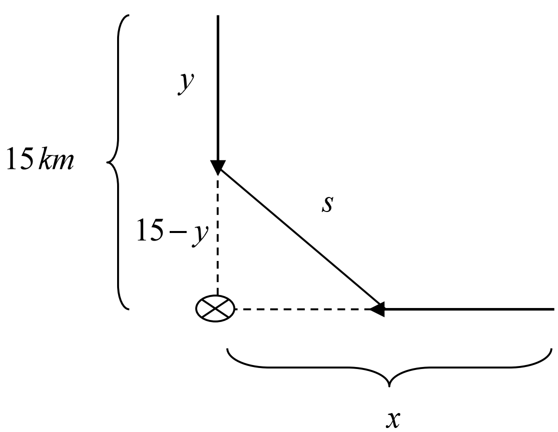
$$t=\frac{1}{29}\text{ hr}\approx 2.1\text{min}$$

$$s=\sqrt{(2-y)^2+x^2}$$

$$=\sqrt{\left[2-50\left(\frac{1}{29}\right)\right]^2+\left[20\left(\frac{1}{29}\right)\right]^2}$$

$$\doteq 0.7728\text{ km}$$

22. At noon a car is driving west at 55 km/h. At the same time, 15 km due north another car is driving south at 85 km/h. At what rate the distance between two cars changing 4 hours later?



After 4hr : $x=55\times4=220\text{km}$

$y=85\times4=340\text{km}$

$$s=\sqrt{x^2+(15-y)^2}$$

$$s=\sqrt{220^2+(15-340)^2}$$

$$s=392.46\text{km}$$

$$s^2=x^2+(15-y)^2$$

$$\cancel{2}s\frac{ds}{dt}=\cancel{2}x\frac{dx}{dt}-\cancel{2}(15-y)\frac{dy}{dt}$$

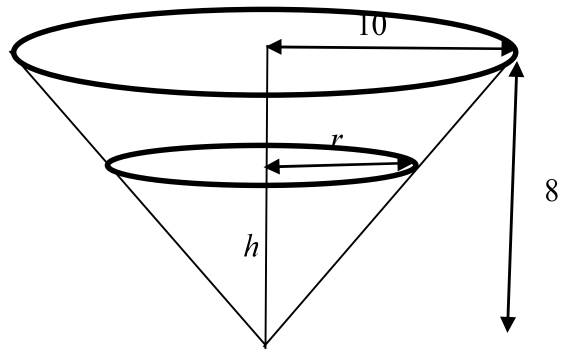
$$(392.46)\frac{ds}{dt}=(220)(55)-(15-340)(85)$$

$$\frac{ds}{dt}=101.2\text{ km/h}$$

23. How fast is the area of a rectangle changing if one side is 10 cm long and is increasing at a rate of 2 cm/s and the other side is 8 cm long and is decreasing at a rate of 3 cm/s ?

$$A = xy$$
$$\frac{dA}{dt} = y\frac{dx}{dt} + x\frac{dy}{dt}$$
$$\frac{dA}{dt} = 10(2) + 8(-3)$$
$$= -4\ cm/s$$

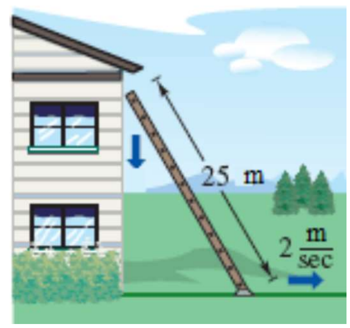
24. A water tank is in the shape of an inverted right circular cone with top radius 10 m and depth 8 m. Water is flowing in at a rate of 0.1 m^3 /min. How fast is the depth of water in the tank increasing 3 minutes later?



$$V = \frac{1}{3}\pi r^2 h$$
$$\frac{dV}{dt} = 0.1 m^3/\min$$
$$\frac{dh}{dt} = ?\ \text{when } t = 3\min$$
$$\frac{10}{r} = \frac{8}{h} \to r = \frac{5}{4}h$$
$$\text{after 3min. } V = 3 \times 0.1$$
$$= 0.3\ m^3$$
$$0.3 = \frac{1}{3}\pi\left(\frac{5}{4}h\right)^2 h$$
$$\frac{3}{10} = \frac{25}{48}\pi h^3$$
$$\boxed{h \doteq 0.57m}$$

$$\frac{dv}{dt} = \frac{25}{16}\pi h^2 \frac{dh}{dt}$$
$$0.1 = \frac{25}{16}\pi(0.57)^2 \frac{dh}{dt}$$
$$\frac{dh}{dt} = 0.062\ m/h$$

25. A ladder 25 m long is leaning against a house. The base of the ladder is pulled away from the wall at a rate of 2 meter per second. How fast is the top of the ladder moving down the wall when the base of the ladder is 12 meter from the wall? Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 m from the wall.



$x^2 + y^2 = 25^2$

When $x = 12\text{m}$

$y = \sqrt{25^2 - 12^2}$

$= 21.9\text{m}$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$(12)(2) + (21.9)\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -1.1\text{m/s}$

$A = \frac{1}{2}xy \;,\quad x^2 + y^2 = 625$

when $x = 7\text{m}$, $y = y = \sqrt{25^2 - 7^2} = 24$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$7(2) + (24)\left(\frac{dy}{dt}\right) = 0$

$\frac{dy}{dt} = -\frac{7}{12}\text{m/s}$

$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$

$= \frac{1}{2}\left(7\left(\frac{-7}{12}\right) + 24(2)\right)$

$= 21.96\text{ m/s}$